Dynamics of a Bright Soliton in Bose-Einstein Condensates with Time-Dependent Atomic Scattering Length in an Expulsive Parabolic Potential

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We present a family of exact solutions of the one-dimensional nonlinear Schrödinger equation which describes the dynamics of a bright soliton in Bose-Einstein condensates with the time-dependent interatomic interaction in an expulsive parabolic potential. Our results show that, under a safe range of parameters, the bright soliton can be compressed into very high local matter densities by increasing the absolute value of the atomic scattering length, which can provide an experimental tool for investigating the range of validity of the one-dimensional Gross-Pitaevskii equation. We also find that the number of atoms in the bright soliton keeps dynamic stability: a time-periodic atomic exchange is formed between the bright soliton and the background.

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With the experimental observation and theoretical studies of Bose-Einstein condensates (BECs) [1], there has been intense interest in the nonlinear excitations of the atomic matter waves, such as dark [2] and bright solitons [3–6]. It is believed that atomic matter bright solitons are of primary importance for developing concrete applications of BECs in the future, so it is of interest to develop a new technique which allows us to construct a particular soliton with the assumed peak matter density. One possibility is to vary the interatomic interaction by means of external magnetic fields. Recent experiments have demonstrated that the variation of the effective scattering length, even including its sign, can be achieved by utilizing the so-called Feshbach resonance [7]. This offers a good opportunity for manipulation of atomic matter waves and nonlinear excitations in BECs [8,9]. In Ref. [10], it has been demonstrated that the variation of nonlinearity of the Gross-Pitaevskii (GP) equation via Feshbach resonance provides a powerful tool for controlling the generation of bright and dark soliton trains starting from periodic waves. Besides, a sinusoidal variation of the scattering length has also been used to form patterns such as Faraday waves [11], or as a means to maintain BECs without an external trap in two dimensions [12].

In this Letter, we present a thorough analysis of the dynamics of a bright soliton of BECs with time-varying atomic scattering length in an expulsive parabolic potential. Our study is greatly facilitated by the so-called Darboux transformation [13], by which we can directly construct the exact solutions of a 1D nonlinear Schrödinger equation (NLSE). Under a safe range of parameters, the bright soliton in BECs can be compressed into very high local matter densities by increasing the absolute value of atomic scattering length with Feshbach resonance. During the compression of the bright soliton in BECs, the number of atoms in the bright soliton keeps dynamic stability and the exchange of the atoms between the bright soliton and the background of density exists.

At the mean-field level, the GP equation governs the evolution of the macroscopic wave function of BECs. In the physically important case of the cigar-shaped BECs, it is reasonable to reduce the GP equation into a 1D NLSE [5,14,15],

$$i\frac{\partial\psi(x,t)}{\partial t}+\frac{\partial^2\psi(x,t)}{\partial x^2}+2a(t)|\psi(x,t)|^2\psi(x,t)+\frac{1}{4}\lambda^2x^2\psi(x,t)=0. \quad (1)$$

In Eq. (1), time t and coordinate x are measured in units $2/\omega_\perp$ and $a_\perp$, where $a_\perp=(\hbar/m\omega_\perp)^{1/2}$ and $a_0=(\hbar/m\omega_0)^{1/2}$ are linear oscillator lengths in the transverse and cigar-axis directions, respectively. $\omega_\perp$ and ω_0 are corresponding harmonic oscillator frequencies, m is the atomic mass, and $\lambda=2|\omega_0|/\omega_\perp\ll 1$. The Feshbach-managed nonlinear coefficient reads $a(t)=|a_s(t)|/a_B=g_0\exp(\lambda t)$ (a_B is the Bohr radius) [16,17]. The normalized macroscopic wave function $\psi(x,t)$ is connected to the original order parameter $\Psi(\mathbf{r},t)$ as follows:

$$\Psi(\mathbf{r},t)=\frac{1}{\sqrt{2\pi a_B}a_\perp}\psi\left(\frac{x}{a_\perp},\frac{\omega_\perp t}{2}\right)\times\exp\left(-i\omega_\perp t-\frac{y^2+z^2}{2a_\perp}\right). \quad (2)$$

From the viewpoint of stability, 3D and 1D equations are very different. For a true 1D system, one does not expect the collapse of the system when increasing number of atoms [5]. However, it happens that a realistic 1D limit is not a true 1D system, with the density of particles still increasing due to the strong restoring forces in the perpendicular directions. To avoid the collapse of the bright soliton [18], we must restrict our study of BECs to a safe

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range of parameters, in which the system becomes effectively 1D; i.e., the energy of two body interactions is much less than the kinetic energy in the transverse direction [15]: $\epsilon^2 = a_\perp/\xi^2 \sim N|a_s|/a_0 \ll 1$ (ξ is the healing length). The bright soliton in BECs has been created with the parameters of $N \approx \times 10^3$, $\omega_\perp = 2\pi \times 700$ Hz and $\omega_0 = 2\pi \times 7$ Hz, and $a_{\rm final} = -4a_B$ for ^{7}Li [4], which provides a safe range of parameters. With the same experimental conditions in Ref. [4] and $a_s(t=0) = -0.25a_B$, we can calculate $\epsilon^2 = a_\perp/\xi^2 \sim N|a_s|/a_0 = 9.5 \times 10^{-3} \ll 1$. Then, the scattering length is increased in the form of $a(t) = g_0 \exp(\lambda t)$. After at least up to 50 dimensionless units of time, the absolute value of the atomic scattering length turns to $|a_s(t)| = 0.8a_B$, corresponding to $\epsilon^2 = a_\perp/\xi^2 \sim N|a_s|/a_0 = 3 \times 10^{-2} \ll 1$. Under the above conditions, the system is effectively 1D. So a safe range of parameters can be described as follows: (i) with the same experimental conditions in Ref. [4]; (ii) ramp up the scattering length in the form of $a(t) = g_0 \exp(\lambda t)$ within 50 dimensionless units of time. We also have to specify the terminology long-time dynamics. A unity of time, $\Delta t = 1$ in the dimensionless variables, corresponds to $2/\omega_\perp$ real seconds. This means, for example, that for a BEC in a trap with transversal size of the order of $a_\perp \approx 1.4$ μm a unity of the dimensionless time corresponds to 4.5×10^{-4} s. The lifetime of a BEC in today's experiments is of the order of 1 s, which is about 220 in our dimensionless units.

The so-called "seed" solution of Eq. (1) can be chosen as follows:

$$\psi_0(x,t) = A_c \exp\left[\frac{\lambda t}{2} + i\varphi_c\right], \tag{3}$$

where $\varphi_c = k_0 x \exp(\lambda t) - \frac{\lambda x^2}{4} + \{(2g_0A_c^2 - k_0^2) \times [\exp(2\lambda t) - 1]/2\lambda\}$, and A_c and k_0 are the arbitrary real constants. We perform the Darboux transformation [19] $\psi_1 = \psi_0 + \frac{2}{\sqrt{g_0}}[(\zeta + \bar{\zeta})\phi_1\bar{\phi}_2/\phi^T\bar{\phi}]\exp(-\lambda t/2 - i\lambda x^2/4)$ to obtain the new solution of Eq. (1) by taking Eq. (3) as the seed. Then we obtain the exact solution of Eq. (1) as follows:

$$\psi = \left[A_c + A_s \frac{(\gamma \cosh\theta + \cos\varphi) + i(\alpha \sinh\theta + \beta \sin\varphi)}{\cosh\theta + \gamma\cos\varphi}\right] \times \exp\left(\frac{\lambda t}{2} + i\varphi_c\right), \tag{4}$$

where

$$\begin{aligned} \theta &= -\frac{[(k_0 + k_s)\Delta_R - \sqrt{g_0}A_s\Delta_I][\exp(2\lambda t) - 1]}{2\lambda} + \Delta_R x \exp(\lambda t), \\ \varphi &= -\frac{[(k_0 + k_s)\Delta_I + \sqrt{g_0}A_s\Delta_R][\exp(2\lambda t) - 1]}{2\lambda} + \Delta_I x \exp(\lambda t), \end{aligned} \tag{5}$$

and

$$\alpha = \frac{\sqrt{g_0}A_c(k_0 - k_s + \Delta_I)}{\Lambda}, \qquad \beta = 1 - \frac{2g_0A_c^2}{\Lambda}, \qquad \gamma = \frac{\sqrt{g_0}A_c(\Delta_R - \sqrt{g_0}A_s)}{\Lambda}, \tag{6}$$

with

$$\begin{aligned} \Delta &= \sqrt{[-\sqrt{g_0}A_s + i(k_s - k_0)]^2 - 4g_0A_c^2} \equiv \Delta_R + i\Delta_I, \\ \Lambda &= g_0A_c^2 + \frac{(\Delta_R - \sqrt{g_0}A_s^2)}{4} + \frac{(k_s - k_0 + \Delta_I)^2}{4}, \end{aligned} \tag{7}$$

where k_s is the arbitrary real constant. On the one hand, when $A_c = k_0 = 0$, Eq. (4) reduces to the well-known soliton solution: $\psi_s = A_s \mathrm{sech}\theta_s \exp(\lambda t/2 + i\varphi_s)$, where $\theta_s = -\sqrt{g_0}\exp(\lambda t)A_s x + \sqrt{g_0}k_sA_s[\exp(2\lambda t) - 1]/\lambda$ and $\varphi_s = \varphi_c - g_0A_c^2[\exp(2\lambda t) - 1]/2\lambda$. On the other hand, when the amplitude of the soliton vanishes ($A_s = 0$), Eq. (4) reduces to Eq. (3). Thus, Eq. (4) represents a bright soliton embedded in the background. Considering the dynamics of the bright soliton in the background, the length $2L$ of the spatial background must be very large compared to the scale of the soliton. In the real experiment [3], the length of the background of BECs can reach at least $2L = 370$ μm. At the same time, in Fig. 1, the width of the bright soliton is about $2l = 2 \times 1.4$ μm $= 2.8$ μm [a unity of coordinate, $\Delta x = 1$ in the dimensionless variables, corresponds to $a_\perp = (\hbar/m\omega_\perp)^{1/2} = 1.4$ μm]. So we indeed have $l \ll L$, a necessary condition for realizing our soliton in experiment.

By utilizing the properties of Eq. (4), we demonstrate that the manipulation of the scattering length can be used to compress a bright soliton of BECs into an assumed peak matter density. It has been reported that an abrupt change of the scattering length can lead to the splitting of the soliton by generating the new solitons. The fragmentation of the soliton obviously decreases the numbers of atoms of

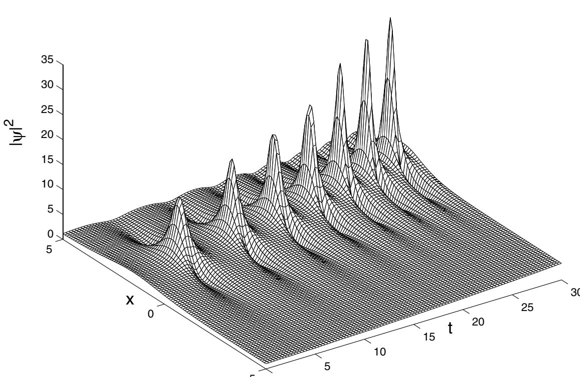

FIG. 1. The dynamics of the Feshbach resonance managed soliton in the expulsive parabolic potential given by Eq. (8). The parameters are given as follows: $\lambda = 2 \times 10^{-2}$, $g_0 = 0.25$, $A_c = 1$, $A_s = 2.4$, and $k_0 = 0.03$.

the original soliton, which is undesirable for application [20]. However, in Eq. (4), the change of the scattering length preserves the soliton from splitting new ones. For simplicity, we assume $k_0 = k_s$ in Eq. (7) and consider only the case of $A_s^2 > 4A_c^2$, for in the case of $A_s^2 < 4A_c^2$, a small perturbation for Eq. (4) may lead to the modulation instability [21]. With the conditions above, Eq. (4) can be deduced into the following form:

$$\psi = \left[-A_c + \delta_2 \frac{\delta_2 \cos\varphi - iA_s \sin\varphi}{A_s \cosh\theta - 2A_c \cos\varphi} \right] \times \exp\left(\frac{\lambda t}{2} + i\varphi_c\right), \tag{8}$$

where

$$\begin{aligned} \theta &= \sqrt{g_0}\delta_2 x \exp(\lambda t) - \frac{\sqrt{g_0} k_0 \delta_2 [\exp(2\lambda t) - 1]}{\lambda}, \\ \varphi &= -\frac{g_0 A_s \delta_2 [\exp(2\lambda t) - 1]}{2\lambda}, \quad \delta_2 = \sqrt{A_s^2 - 4A_c^2}. \end{aligned} \tag{9}$$

For a better understanding, we plot Eq. (8) in Fig. 1, which shows the dynamics of the Feshbach resonance managed bright soliton in the expulsive parabolic potential. As one can see from Fig. 1, with the increasing absolute value of the scattering length, the bright soliton has an increase in the peaking value and a compression in the width. As a result, we can get a bright soliton with the assumed peak matter density. It is interesting to observe that in the expulsive parabolic potential, the bright soliton is set into motion and propagates in the longitudinal direction, instead of oscillating in an attractive parabolic potential. The possibility of compressing the soliton of BECs into an assumed peak matter density could provide an experimental tool for investigating the range of validity of the 1D GP equation. Since the quasi-1D GP equation applies only for low densities, it would indeed be interesting to see how far one can compress a soliton in a real experiment by increasing the absolute value of the scattering length.

Inspired by two experiments [3,4], we can design the compression of a bright soliton in BECs with the following steps: (i) Create a bright soliton in BECs with the parameters of $N \approx \times 10^3$, $\omega_\perp = 2\pi \times 700$ Hz and $\omega_0 = 2\pi \times 7$ Hz, and $a_s = -0.25 a_B$ for ^{7}Li. The main effect of this expulsive term is that the center of the BECs accelerates along the longitudinal direction. (ii) Under the safe range of parameters discussed above, ramp up the absolute value of the scattering length according to $a(t) = g_0 \exp(\lambda t)$ due to Feshbach resonance, where $\lambda = 2|\omega_0|/\omega_\perp = 2 \times 10^{-2}$ is a very small value. A unity of time, $\Delta t = 1$ in the dimensionless variables, corresponds to $2/\omega_\perp = 4.5 \times 10^{-4}$ real seconds. (iii) After at least up to 50 dimensionless units of time, the absolute value of the atomic scattering length turns to $|a_s(t)| = 0.8 a_B$, which is less than $|a_{\text{final}}| = 4a_B$. This means that during the process of compressing the bright soliton the stability of soliton and the validity of 1D approximation can be kept as displayed in Fig. 1. Therefore, the phenomena discussed in this Letter should be observable within the current experimental capability.

Furthermore, based on Eq. (8), we find that when $\sinh\theta = 0$ the peak matter density of bright soliton arrives at the maximum

$$|\psi|^2 = \exp(\lambda t)\left(A_c^2 + \frac{\delta_2 A_s}{A_s - 2A_c \cos\varphi}\right), \tag{10}$$

and that when $\cosh\theta = \frac{A_s}{A_c \cos\varphi} - \frac{A_c \cos\varphi}{A_s}$ the peak matter density of bright soliton arrives at the minimum

$$|\psi|^2 = \exp(\lambda t)\left(A_c^2 - \frac{A_c^2 \delta_2^2 \cos^2\varphi}{A_s^2 - 4A_c^2 \cos^2\varphi}\right). \tag{11}$$

This means that the bright soliton can only be squeezed into the assumed peak matter density between the minimum and maximum values. In order to investigate the stability of the bright soliton against the variation of the scattering length in the expulsive parabolic potential, we obtain

$$\lim_{L\to\infty} \int_{-L}^{+L} [|\psi(x,t)|^2 - |\psi(\pm L, t)|^2] dx = \frac{2\delta_2}{\sqrt{g_0}}, \tag{12}$$

which is the exact number of the atoms in the bright soliton against the background described by Eq. (8). This indicates that during the process of the compression of the bright soliton the number of atoms in the bright soliton keeps invariant. In contrast, the quantity

$$\begin{aligned} \kappa &= \lim_{L\to\infty} \int_{-L}^{+L} |\psi(x,t) - \psi(\pm L, t)|^2 dx \\ &= \frac{2\delta_2}{\sqrt{g_0}}(1 + A_c M \cos\varphi), \end{aligned} \tag{13}$$

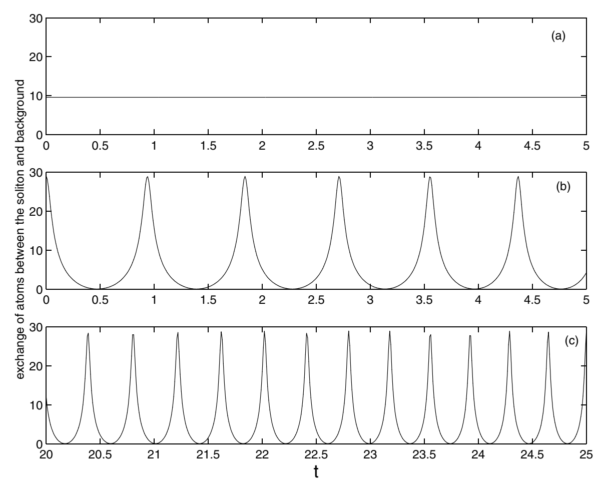

FIG. 2. Time-periodic atomic exchange between the bright soliton and the background given by Eq. (13). The range of time is (a),(b) $t = [0, 5]$ and (c) $t = [20, 25]$. The parameters are given as follows: $\lambda = 0.02$, $g_0 = 1$, $A_s = 4.8$, (a) $A_c = 0$, and (b),(c) $A_c = 2.3$.

where

$$M = \frac{4\arctan(\sqrt{A_s + 2A_c\cos\varphi}/\sqrt{A_s - 2A_c\cos\varphi})}{\sqrt{A_s^2 - 4A_c^2\cos^2\varphi}} \quad (14)$$

counts the number of atoms in both the bright soliton and background under the condition of $\psi(\pm L, t) \neq 0$. Equation (13) displays that a time-periodic atomic exchange is formed between the bright soliton and the background. As shown in Fig. 2(a), in the case of zero background, i.e., $A_c = 0$, there will not be the exchange of atoms. However, in the case of nonzero background as shown in Figs. 2(b) and 2(c), the exchange of atoms between the bright soliton and the background becomes quicker when increasing the absolute value of the scattering length. The conclusion can be made that the number of atoms in the bright soliton in BECs keeps dynamic stability against the variation of the scattering length in the expulsive parabolic potential. The consideration provided above implies that we should construct the bright soliton in BECs on the background. A question arises about the possibility of the creation of such a state experimentally. We notice that Eq. (8) will take the particular form at the time $t_0 = \frac{1}{2\lambda}\ln[-\frac{\lambda(4n+1)\pi}{A_s\delta_2 g_0} + 1]$, $n = -1, -2, -3\ldots$, as follows:

$$\begin{aligned}\psi(x,t) &= -A_c\exp\left(\frac{\lambda t_0}{2}\right)\exp(i\varphi_c) \\ &\quad - i\delta_2\exp\left(\frac{\lambda t_0}{2}\right)\mathrm{sech}\theta\exp(i\varphi_c), \\ &= -\psi_0(x,t_0) - \psi_{\mathrm{soliton}}(x,t_0), \end{aligned} \quad (15)$$

which is the linear combination of Eq. (3) and a bright soliton. So Eq. (15) means that Eq. (8) can be generated by coherently adding a bright soliton into the background of density described by Eq. (3).

In conclusion, we present a family of exact solutions of the nonlinear Schrödinger equation with the time-varying nonlinear coefficient in the expulsive parabolic potential. Our results describe the dynamics of the Feshbach resonance managed bright soliton of BECs in an expulsive parabolic potential. Furthermore, under a safe range of parameters, it is possible to squeeze a bright soliton of BECs into the assumed peak matter density, which can provide an experimental tool for investigating the range of validity of the 1D GP equation. We also find that the number of atoms in the bright soliton keeps dynamic stability: the exchange of atoms between the bright soliton and the background becomes quicker when increasing the absolute value of the scattering length. Recent developments of controlling the scattering length in the experiments allow for the experimental investigation of our prediction in the future.

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[1] F. Dalfovo *et al.*, Rev. Mod. Phys. **71**, 463 (1999).
[2] S. Burger *et al.*, Phys. Rev. Lett. **83**, 5198 (1999); J. Denschlag *et al.*, Science **287**, 97 (2000); Th. Busch and J. R. Anglin, Phys. Rev. Lett. **84**, 2298 (2000); C. K. Law *et al.*, Phys. Rev. Lett. **85**, 1598 (2000); B. P. Anderson *et al.*, Phys. Rev. Lett. **86**, 2926 (2001); Biao Wu *et al.*, Phys. Rev. Lett. **88**, 034101 (2002).
[3] K. E. Strecker *et al.*, Nature (London) **417**, 150 (2002).
[4] L. Khaykovich *et al.*, Science **296**, 1290 (2002).
[5] V. M. Perez-Garcia *et al.*, Phys. Rev. A **57**, 3837 (1998); G. Fibich *et al.*, Physica (Amsterdam) **175D**, 96 (2003); G. Fibich *et al.*, Nonlinearity **16**, 1809 (2003); G. Fibich *et al.*, Phys. Rev. Lett. **90**, 203902 (2003).
[6] P. G. Kevrekidis *et al.*, Phys. Rev. Lett. **90**, 230401 (2003); Shun-Jin Wang *et al.*, Phys. Rev. A **68**, 015601 (2003).
[7] J. L. Roberts *et al.*, Phys. Rev. Lett. **81**, 5109 (1998); J. Stenger *et al.*, Phys. Rev. Lett. **82**, 2422 (1999).
[8] D. E. Pelinovsky *et al.*, Phys. Rev. Lett. **91**, 240201 (2004).
[9] Z. X. Liang *et al.* (unpublished).
[10] F. Kh. Abdullaev *et al.*, Phys. Rev. Lett. **90**, 230402 (2003).
[11] K. Staliunas *et al.*, Phys. Rev. Lett. **89**, 210406 (2002).
[12] H. Saito *et al.*, Phys. Rev. Lett. **90**, 040403 (2003); F. Kh. Abdullaev *et al.*, Phys. Rev. A **67**, 013605 (2003).
[13] V. B. Matveev and M. A. Salli, *Darboux Transformations and Solitons*, Springer Series in Nonlinear Dynamics (Springer, Berlin, 1991).
[14] P. G. Kevrekidis and D. J. Frantzeskakis, Mod. Phys. Lett. B **18**, 173 (2004).
[15] V. A. Brazhnyi and V. V. Konotop, Mod. Phys. Lett. B **18**, 627 (2004), and references therein.
[16] V. M. Perez-Garcia *et al.*, Phys. Rev. Lett. **92**, 220403 (2004).
[17] Because of $\lambda \ll 1$, the nonlinear coefficient $a(t) = g_0 e^{\lambda t}$ can be expressed as $a(t) = g_0[1 + \lambda t + 0^2(\lambda t)]$.
[18] G. Fibich *et al.*, SIAM J. Appl. Math. **60**, 183 (1999); G. Fibich *et al.*, SIAM J. Appl. Math. **61**, 1680 (2001); G. Fibich *et al.*, SIAM J. Appl. Math. **62**, 1437 (2002).
[19] Considering the following UV pair, $\phi_x = U\phi$, $\phi_t = V\phi$, and $\phi = (\phi_1, \phi_2)^T$, where $U = \zeta J + P$, $V = 2i\zeta^2 J + \lambda x\zeta J + 2i\zeta P + W$, with

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & \sqrt{g_0}Q \\ -\sqrt{g_0}\bar{Q} & 0 \end{pmatrix},$$

$$W = \begin{pmatrix} ig_0|Q|^2 & -\sqrt{g_0}\lambda xQ + i\sqrt{g_0}Q_x \\ \sqrt{g_0}\lambda x\bar{Q} + i\sqrt{g_0}\bar{Q}_x & -ig_0|Q|^2 \end{pmatrix}.$$

Here the overbar denotes the complex conjugate and $\psi = Q\exp(-i\lambda x^2/4 - \lambda t/2)$. From the compatibility condition $U_t - V_x + [U, V] = 0$, one can derive Eq. (1) in the case of $\zeta = \xi_0\exp(\lambda t)$.
[20] L. D. Carr and Y. Castin, Phys. Rev. A **66**, 063602 (2002).
[21] L. Salasnich *et al.*, Phys. Rev. Lett. **91**, 080405 (2003); G. Fibich *et al.*, Phys. Lett. A **239**, 167 (1998).