Optically trapped quasi-two-dimensional Bose gases in a random environment: Quantum fluctuations and superfluid density

Kezhao Zhou (周可召),[1] Ying Hu (胡颖),[2] Zhaoxin Liang (梁兆新),[1,*] and Zhidong Zhang (张志东)[1]
[1] *Shenyang National Laboratory for Materials Science, Institute of Metal Research, and International Centre for Materials Physics, Chinese Academy of Sciences, 72 Wenhua Road, Shenyang 110016, People's Republic of China*
[2] *Department of Physics, Centre for Nonlinear Studies, and The Beijing-Hong Kong-Singapore Joint Centre for Nonlinear and Complex Systems (Hong Kong), Hong Kong Baptist University, Kowloon Tong, Hong Kong, China*
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We investigate a dilute Bose gas confined in a tight one-dimensional (1D) optical lattice plus a superimposed random potential at zero temperature. Accordingly, the ground-state energy, quantum depletion, and superfluid density are calculated. The presence of the lattice introduces a crossover to the quasi-two-dimensional (2D) regime, where we analyze asymptotically the 2D behavior of the system, particularly the effects of disorder. We thereby offer an analytical expression for the ground-state energy of a purely 2D Bose gas in a random potential. The obtained disorder-induced normal fluid density n_n and quantum depletion n_d both exhibit a characteristic $1/\ln(1/n_{2\mathrm{D}}a_{2\mathrm{D}}^2)$ dependence. Their ratio n_n/n_d increases to 2 compared to the familiar 4/3 in lattice-free three-dimensional (3D) geometry, signifying a more pronounced contrast between superfluidity and Bose-Einstein condensation in low dimensions. The conditions for possible experimental realization of our scenario are also proposed.

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I. INTRODUCTION

The effect of dimensionality of a bosonic system on the presence and nature of the Bose-Einstein condensation (BEC) as well as on the superfluid phase transition has received long-standing interest both experimentally and theoretically [1,2]. The physics at low dimensions exhibit fundamental differences from that in three-dimensions (3D). In particular, the strong long-range phase fluctuations typical of low-dimensional bosonic systems usually inhibit the formation of long-range order, which, on the other hand, characterizes the 3D BEC and corresponding phase transition at low temperature [3].

Earlier work on low-dimensional bosonic systems [2] has culminated in, particularly in the uniform two-dimensional (2D) case, two important theoretical discoveries. The first is that in 2D a true condensate can only occur at $T = 0$ and its absence at finite temperature follows from the Bogoliubov k^{-2} [4] or Hohenberg-Mermin-Wagner (BHMW) theorem [5,6]. On the other hand, a superfluid phase transition has been proven to exist at sufficiently low temperature in 2D [7,8]. However, according to Kosterlitz and Thouless (KT) [9], such a transition is associated with the unbinding of vortex pairs or quasi-long-range order, in contrast to the 3D phase transition that features a long-range order parameter. Below the KT transition temperature, a 2D Bose gas (liquid) is characterized by the presence of a "quasicondensate" [10,11].

The remarkable experimental progress with ultracold atomic gases, especially in the cooling and confining of cold atomic gases in traps with controllable geometry and dimension, has significantly stimulated new interest in low-dimensional systems [12,13]. Tight confinement in one or two directions considerably affects the properties of Bose gases such as collisions and phase fluctuations [14], introducing a crossover to the quasi-low-dimensional regime. As such, quasi-2D quantum degenerate Bose gases have been experimentally produced both in single "pancake" traps and at the nodes of a one-dimensional (1D) optical lattice [15].

However, these marginal 2D Bose gases are qualitatively different from the corresponding infinite ones. Along this line, Petrov *et al.* [14] pointed out that the presence of the trapping potential suppresses long-range thermal fluctuations and that in a quasi-2D system a true condensate can exist within a wide parameter range. Moreover, Fischer [16] has obtained in a marginal 2D case a model-independent geometrical equivalence of the BHMW theorem.

Compared with harmonically trapped systems, optically trapped Bose gases allow more experimental controllability with tunable interatomic interactions, tunneling amplitudes between adjacent sites, atom filling fractions, and lattice dimensionality [12,17], thereby presenting a more useful testing ground for theoretical ideas in studying low-dimensional systems in novel conditions. On the other hand, disorder has been observed to cause a dramatic influence on a BEC and has attracted great interest recently [18,19]. In view of the availability to control a 1D optical lattice and external randomness, therefore, one especially appealing direction of the investigation consists in studying the effect of external randomness on a Bose gas trapped in a 1D optical lattice.

In this paper, we investigate the ground-state properties and superfluidity of a 1D-optical-lattice-trapped Bose gas in a random environment at $T = 0$. Capitalizing on the characteristic lattice-induced 3D to quasi-2D dimensional crossover, we analyze the effects of disorder in the asymptotic 2D regime. The present work is composed of two parts. In the first part, we calculate the ground-state energy and quantum depletion for the model system using the path-integral approach within the Bogoliubov approximation. A discussion on the dimensional crossover property in a random potential is presented. In particular, our results in the quasi-2D regime

*zhxliang@gmail.com

with varnishing disorder are in good agreement with that of a homogeneous 2D Bose gas at $T = 0$ [2,20,21]. We suggest, therefore, that our result gives the analytical expression for the ground-state energy of a uniform 2D dilute Bose gas in the presence of weak disorder. In the second part, we calculate the disorder and lattice-induced normal fluid density n_n at $T = 0$. Our results in the anisotropic 3D regime reproduce the well-known ratio $n_n/n_d = 4/3$ [22,23] with n_d being the quantum depletion due to disorder. Whereas, in the quasi-2D regime, n_n exhibits a $1/\ln(1/n_{2D}a_{2D}^2)$ dependence unique to a 2D system and the ratio becomes asymptotically $n_n/n_d = 2$, indicating a more pronounced contrast between superfluidity and BEC in low dimensions.

The outline of the paper is as follows. In Sec. II, we introduce the grand-canonical partition function for a dilute Bose gas in the presence of a 1D optical lattice and weak disorder at $T = 0$. Accordingly, the analytical expressions for the ground-state energy and quantum depletion are derived. Section III presents a detailed discussion on the dimensional crossover in the ground-state properties induced by a 1D optical lattice. The effects of disorder in the crossover regimes are analyzed. In Sec. IV, we calculate the superfluid density and study its behavior in, respectively, the 3D and quasi-2D regime. Finally, we summarize our results in Sec. V and propose possible experimental scenarios.

II. BOSE GASES IN THE PRESENCE OF A 1D OPTICAL LATTICE AND WEAK DISORDER

A. Path-integral approach

Our starting point is the grand-canonical partition function of a 3D weakly interacting dilute Bose gas [10] in the presence of a 1D optical lattice and weak disorder

$$Z = \int D[\psi^*,\psi]e^{-\frac{S[\psi^*,\psi]}{\hbar}}, \tag{1}$$

where the action functional $S[\psi^*,\psi]$ reads

$$S[\psi^*,\psi] = \int_0^{\hbar\beta} d\tau \int d\mathbf{r}\psi^*(\mathbf{r},\tau)\left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} - \mu + V_{\rm opt}(\mathbf{r}) + V_{\rm ran}(\mathbf{r}) + \frac{g_e}{2}|\psi(\mathbf{r},\tau)|^2\right]\psi(\mathbf{r},\tau). \tag{2}$$

In Eqs. (1) and (2), $[\psi^*(\mathbf{r},\tau),\psi(\mathbf{r},\tau)]$ collectively denote the complex functions of space and imaginary time τ, $\beta = 1/k_BT$ with k_B being the Boltzmann constant and T being the temperature, μ is the chemical potential, and g_e is the effective two-body coupling constant in the presence of a 1D optical lattice. The $V_{\rm opt}(\mathbf{r})$ and $V_{\rm ran}(\mathbf{r})$, respectively, represent the 1D optical lattice and external random potential.

The optical potential $V_{\rm opt}(\mathbf{r})$ in Eq. (2) is given by

$$V_{\rm opt}(\mathbf{r}) = sE_R\sin^2(q_Bz), \tag{3}$$

where s is a dimensionless factor labeled by the intensity of a laser beam and $E_R = \hbar^2q_B^2/2m$ is the recoil energy with $\hbar q_B$ being the Bragg momentum. The lattice period is fixed by $q_B = \pi/d$ with d being the lattice spacing. Atoms are unconfined in the x-y plane.

Disorder $V_{\rm ran}(\mathbf{r})$ in Eq. (2) is produced by the random potential associated with quenched impurities [22–24]

$$V_{\rm dis}(\mathbf{r}) = \sum_{i=1}^{N_{\rm imp}} v(|\mathbf{r}-\mathbf{r_i}|), \tag{4}$$

with $v(\mathbf{r})$ describing the two-body interaction between bosons and impurities, $\mathbf{r}_i$ being the randomly distributed positions of impurities, and $N_{\rm imp}$ counting the number of $\mathbf{r}_i$. Here, we restrict ourselves to the conditions of a dilute BEC system in the presence of a very small concentration of disorder. Thereby, $v(\mathbf{r})$ can be approximated by an effective pseudopotential in the form $v(\mathbf{r}) = g_{\rm imp}\,\delta(\mathbf{r})$ [22], with $g_{\rm imp}$ being the effective coupling constant of an impurity-boson pair confined in a 1D optical lattice.

It is important to mention that the tight confinement in the direction of the optical lattice considerably influences the value of the effective coupling constant [14,25] in Eq. (2). Particularly, in the presence of the optical lattice, g_e generally exhibits dependence on the density and lattice parameter [26], in marked contrast to a free 3D Bose gas where $g_{3D} = 4\pi\hbar^2a_{3D}/m$ with a_{3D} being the 3D scattering length. For formulation clarity, however, in the following we shall use g_e and $g_{\rm imp}$ for notational convenience while leaving aside their specific expressions to obtain general expressions for the ground-state energy and quantum depletion. An analysis of the lattice-renormalized effective coupling constant will be given in Sec. IV.

B. Beyond-mean-field ground-state energy and quantum depletion

In what follows, we focus on the situation where the optical lattice is strong enough to create many separated wells that give rise to an array of condensates, while full coherence is still assured by the quantum tunneling. By this assumption, one can refer to n_0 as the condensate density and neglect the Mott insulator phase transition. We also suppose disorder is sufficiently weak. Under these conditions, one is able to investigate the ground-state properties of the model system using Bogoliubov's theory [3].

We shall restrict ourselves to the case where s is relatively large that the interwell barriers are significantly higher than the chemical potential μ [27]. We thereby only consider the lowest Bloch band where the condensate, in the tight-binding approximation, can be written in terms of Wannier functions as $\phi_{k_z}(z) = \sum_l e^{ilk_z}w(z-ld)$ where $w(z) = \exp(-z^2/2\sigma^2)/\pi^{1/4}\sigma^{1/2}$ with $d/\sigma \simeq \pi s^{1/4}\exp(-1/4\sqrt{s})$. Expanding the bosonic field variables in Eq. (2) by the expression $\psi(\mathbf{r},\tau) = \sum_{\mathbf{k},n}\psi_{\mathbf{k},n}\phi_{k_z}(z)e^{-i(k_xx+k_yy)}e^{i\omega_n\tau}$ with $\omega_n = 2\pi n/\hbar\beta$ being the bosonic Matsubara frequencies where n are integers, the action Eq. (2) takes the form

$$\frac{S[\psi^*,\psi]}{\hbar\beta V} = \sum_{\mathbf{k},n}\psi^*_{\mathbf{k},n}\left[-i\hbar\omega_n + \varepsilon^0_{\mathbf{k}} - \mu\right]\psi_{\mathbf{k},n} + \frac{\tilde{g}_e}{2}\sum_{\substack{\mathbf{k},\mathbf{k}',\mathbf{q}\\ n,n',m}}\psi^*_{\mathbf{k}+\mathbf{q},n+m}\psi^*_{\mathbf{k}'-\mathbf{q},n'-m}\psi_{\mathbf{k}',n'}\psi_{\mathbf{k},n} + \sum_{\mathbf{k},\mathbf{k}',n}V_{\mathbf{k}-\mathbf{k}'}\psi^*_{\mathbf{k},n}\psi_{\mathbf{k}',n}. \tag{5}$$

Here $\varepsilon_{\mathbf{k}}^0 = (\hbar^2/2m)(k_x^2 + k_y^2) - 2t[1 - \cos(k_z d)]$, with t being the tunneling rate between neighboring wells, is the energy dispersion of the noninteracting model, V is the volume of the system, and $\tilde{g}_e$ is the lattice-renormalized coupling constant given by

$$\tilde{g}_e = g_e \left[d \int_{-d/2}^{d/2} w^4(z)\, dz \right] = g_e \frac{d}{\sqrt{2\pi}\sigma}. \tag{6}$$

In Eq. (5), the V_k is the Fourier transform of $\tilde{V}_{\mathrm{ran}}(\mathbf{r}) = \sum_i \tilde{g}_{\mathrm{imp}} \delta(\mathbf{r} - \mathbf{r}_i)$ with $\tilde{g}_{\mathrm{imp}} = g_{\mathrm{imp}} d/\sqrt{2\pi}\sigma$ being the lattice-renormalized impurity-boson coupling constant [i.e., $V_{\mathbf{k}} = (1/V) \int e^{i\mathbf{k}\mathbf{r}} \tilde{V}_{\mathrm{ran}}(\mathbf{r})\, d\mathbf{r}$]. For simplicity, the external randomness is assumed to be uniformly distributed with density $n_{\mathrm{imp}} = N_{\mathrm{imp}}/V$ and Gaussian correlated [28]. Hence the two basic statistical properties of the disorder are the average value $\langle V_0 \rangle = \tilde{g}_{\mathrm{imp}} n_{\mathrm{imp}}$ and the correlation function $\langle V_{-\mathbf{k}} V_{\mathbf{k}} \rangle = \tilde{g}_{\mathrm{imp}}^2 n_{\mathrm{imp}}/V$. Here the notation $\langle .. \rangle$ stands for the ensemble average over all possible realizations of disorder configurations.

By applying the Bogoliubov theory to the action (5) and proceeding in the standard fashion [3], one obtains the zero-temperature thermodynamic function $\Omega = E_g - V\mu n_0$ with the ground-state energy E_g reading

$$\frac{E_g}{V} = \frac{1}{2}\tilde{g}_e n_0^2 - \frac{1}{2V} \sum_{\mathbf{k}\neq 0} \left(\varepsilon_{\mathbf{k}}^0 + \tilde{g}_e n_0 - E_{\mathbf{k}} \right) + n_0 \left[n_{\mathrm{imp}} \tilde{g}_{\mathrm{imp}} - \frac{n_{\mathrm{imp}} \tilde{g}_{\mathrm{imp}}^2}{V} \sum_{\mathbf{k}\neq 0} \frac{\varepsilon_{\mathbf{k}}^0}{E_{\mathbf{k}}^2} \right]. \tag{7}$$

Here $E_{\mathbf{k}} = \sqrt{(\varepsilon_{\mathbf{k}}^0 - \mu + 2\tilde{g}_e n_0)^2 - \tilde{g}_e^2 n_0^2}$ is the energy spectrum of the elementary excitations and n_0 is the condensate density. In conformity with the general theory, we set $\mu = \tilde{g}_e n_0$ to ensure a gapless quasiparticle spectrum [29].

In the continuum limit, the sum in Eq. (7) is replaced with integrals. To avoid the large-k divergence in the integration over k_x and k_y, however, one must introduce a renormalization of the coupling constant by replacing $\tilde{g}_e \to \tilde{g}_e - (\tilde{g}_e^2/V) \sum_{\mathbf{k}\neq 0} (1/2\varepsilon_{\mathbf{k}}^0)$ and $\tilde{g}_{\mathrm{imp}} \to \tilde{g}_{\mathrm{imp}} - (\tilde{g}_{\mathrm{imp}}^2/V) \sum_{\mathbf{k}\neq 0} (1/2\varepsilon_{\mathbf{k}}^0)$ in Eq. (7). Upon this replacement, one obtains after integration

$$\frac{E_g}{V} = \frac{1}{2}\tilde{g}_e n_0^2 \left\{ (1+\gamma) + \frac{m\tilde{g}_e}{2\pi^2\hbar^2 d} F\left(\frac{2t}{\tilde{g}_e n_0}\right) + \frac{m\tilde{R}\tilde{g}_e}{2\pi\hbar^2 d} \operatorname{arccoth}\left[\left(\frac{2t}{\tilde{g}_e n_0} + 1 \right)^{\frac{1}{2}} \right] \right\}, \tag{8}$$

where the two parameters $\gamma = 2\kappa \tilde{g}_{\mathrm{imp}}/\tilde{g}_e$ with $\kappa = n_{\mathrm{imp}}/n_0$ and

$$\tilde{R} = \frac{n_{\mathrm{imp}}}{n_0} \frac{4\tilde{g}_{\mathrm{imp}}^2}{\tilde{g}_e^2}, \tag{9}$$

characterize the strength of disorder in a 1D optical lattice. In Eq. (8), the function $F(x)$ with the variable $x = 2t/(\tilde{g}_e n_0)$ is defined as

$$F(x) = \frac{(x+1)}{2} \left[(3x+1) \arctan\left(\frac{1}{\sqrt{x}}\right) - 3\sqrt{x} \right] - \frac{\pi}{2} \ln\left[\frac{x}{2x+1+2\sqrt{x(x+1)}} \right] - \pi \operatorname{arcsinh}(\sqrt{x}) + 2\int_0^{\sqrt{x}} \frac{\tan^{-1}(z)}{z} dz. \tag{10}$$

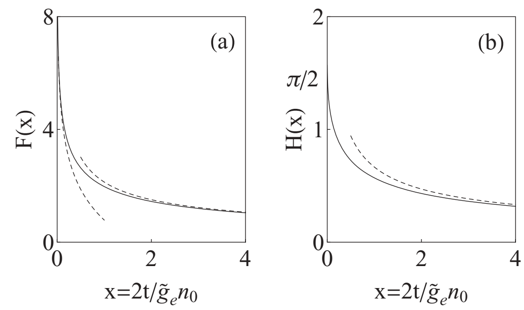

FIG. 1. (a) Scaling function $F(x)$ in Eq. (10) (solid line) and its asymptotic behavior (dashed line). (b) Scaling function $H(x)$ in Eq. (13) (solid line) and its asymptotic behavior (dashed line).

The integration in Eq. (10) can be easily performed numerically and the result is shown in Fig. 1(a). In the ground-state energy Eq. (8), the first two terms give the mean-field contribution modified by an optical lattice and disorder, whereas, the last two terms represent beyond-mean-field corrections, as a consequence of quantum fluctuations, respectively, induced by interatomic interaction and external randomness.

Quantum depletion ($\Delta N = N - N_0$) refers to the average number of atoms with nonzero momentum [3] which can be calculated within the Bogliubov's theory as

$$\Delta N = \sum_{\mathbf{k}\neq 0} \left[\frac{\varepsilon_{\mathbf{k}}^0 + \tilde{g}_e n_0 - E_{\mathbf{k}}}{2E_{\mathbf{k}}} + n_0 n_{\mathrm{imp}} \tilde{g}_{\mathrm{imp}}^2 \frac{(\varepsilon_{\mathbf{k}}^0)^2}{E_{\mathbf{k}}^4} \right]. \tag{11}$$

By replacing the sum with the integral in the continuum limit, one obtains

$$\frac{\Delta N}{N} = \frac{m\tilde{g}_e}{2\pi^2\hbar^2 d} \left[H\left(\frac{2t}{\tilde{g}_e n_0}\right) + \frac{\pi \tilde{R}}{8} \left(1 + \frac{2t}{\tilde{g}_e n_0}\right)^{-\frac{1}{2}} \right], \tag{12}$$

where the function $H(x)$ with $x = 2t/(\tilde{g}_e n_0)$ is defined as

$$H(x) = (x+1) \arctan\left(\frac{1}{\sqrt{x}}\right) - \sqrt{x}. \tag{13}$$

III. DIMENSIONAL CROSSOVER FROM 3D TO QUASI-2D AND 2D REGIMES

At low energies, the physical properties of a dilute Bose gas can be expressed in terms of the two-body scattering amplitude [30]. It has been well established that a tight confinement along one or two directions will considerably affect the scattering properties of atoms, particularly, introducing a dimensional crossover from anisotropic 3D to low-dimensional regimes [14,26].

The two-body scattering problem in the presence of a 1D optical lattice has been analytically investigated in Ref. [26]. For sufficiently deep lattices and chemical potential μ, which is small compared to the interband gap, two distinct regimes can be identified. (i) For $\mu \ll 4t$, where the wave function spreads over many lattice sites, the system retains an anisotropic 3D behavior. In this limit, Eq. (6) takes the limiting form $\tilde{g}_e = \tilde{g}_{3D}$ with

$$\tilde{g}_{3D} = \frac{4\pi\hbar^2 \tilde{a}_{3D}}{m}, \tag{14}$$

with $\tilde{a}_{3D} = a_{3D}d/(\sqrt{2\pi}\sigma)$ being the lattice-renormalized s-wave scattering length. (ii) For $\mu \gg 4t$, the tunneling between wells is negligible and the two interacting bosons are in the ground state of an effective harmonic potential characterized by frequency $\omega_0 = \hbar/m\sigma^2$ and harmonic oscillator length σ. In this limit, the system undergoes a crossover to the quasi-2D regime where the coupling constant is reduced to that in a tight confined harmonic trap $\tilde{g}_e = g_h d$ [14,26,31] where

$$g_h = \frac{2\sqrt{2\pi}\hbar^2}{m}\frac{1}{a_{2D}/a_{3D} + (1/\sqrt{2\pi})\ln\left[1/n_{2D}a_{2D}^2\right]}, \tag{15}$$

with the surface density $n_{2D} = n_0 d$ and the effective 2D scattering length $a_{2D} = \sqrt{\hbar/m\omega_0} = \sigma$ [14]. With decreasing σ, the 2D features in the scattering of two atoms become pronounced [14]; and in the limit $\sigma \ll a$, Eq. (15) becomes independent of the value of a_{3D} and a regime of purely 2D scattering is achieved with Eq. (15) reducing to the coupling constant of a purely 2D Bose gas $g_h \rightarrow g_{2D}$ where

$$g_{2D} = \frac{4\pi\hbar^2}{m}\frac{1}{\ln\left(1/n_{2D}a_{2D}^2\right)}. \tag{16}$$

Here the logarithmic dependence on the gas parameter $n_{2D}a_{2D}^2$ is unique of the 2D geometry.

Taking into account the dimensional crossover in the effective coupling constant, in the following we focus on analyzing the behavior of the ground-state energy in Eq. (8) and quantum depletion in Eq. (12), respectively, in the anisotropic 3D and 2D geometry. In the limit $2t/n_0\tilde{g}_e \gg 1$, corresponding to the anisotropic 3D regime, we find $F(x) \simeq 32/15\sqrt{x}$, as is shown in Fig. 1(a) with the dashed curve. Substitutions of this limiting value in Eq. (8) together with Eq. (14) yield the ground-state energy of an effectively free 3D Bose gas composed of bosons with effective mass $m^* = \hbar^2/2td^2$ and coupling constant $\tilde{g}_{3D}$ [32,33]

$$\frac{E_g}{V} = \frac{1}{2}\tilde{g}_{3D}n_0^2\left[\left(1 + \kappa\frac{\tilde{b}_{3D}}{\tilde{a}_{3D}}\right) + \frac{128}{15}\sqrt{\frac{m^*}{m}}\left(\frac{n_0\tilde{a}_{3D}^3}{\pi}\right)^{1/2} + 4\pi\tilde{R}_{3D}\sqrt{\frac{m^*}{m}}\left(\frac{n_0\tilde{a}_{3D}^3}{\pi}\right)^{1/2}\right]. \tag{17}$$

In Eq. (17), the two characteristic parameters of disorder in Eq. (8), respectively, take their 3D value (i.e., $\gamma = \kappa\tilde{b}_{3D}/\tilde{a}_{3D}$ and $\tilde{R}_{3D} = \kappa\tilde{b}_{3D}^2/\tilde{a}_{3D}^2$) showing the 3D feature of the interaction between the impurity-boson pair. The first term in Eq. (17) represents the mean-field ground-state energy; whereas the remaining terms exhibit the familiar dependence on the effective 3D gas parameter $\sqrt{n_0\tilde{a}_{3D}^3}$, thereby consisting of the generalized Lee-Huang-Yang (LHY) correction [34] to the presence of a 1D optical lattice and weak disorder. Equation (17) bears a formal resemblance with the corresponding result in Ref. [23], which deals with a 2D optical lattice system, consistent with the effective mass theory in the 3D limit where the lattice system is effectively treated as a free gas with effective mass and coupling constant. The main difference is related to the value of the renormalized coupling constant $\tilde{g}_{3D}$ where the renormalization factor is different for various lattice dimensions [32].

In the opposite 2D regime where $2t/\tilde{g}_e n_0 \ll 1$ and $\sigma \ll a$, $F(x)$ exactly approaches a limit $F(x) = \pi/4 - \pi/2\ \ln x$ with $\ln x \simeq \ln(mt/n_{2D}2\pi\hbar^2) + \ln[\ln(1/n_{2D}a_{2D}^2)]$, as shown in Fig. 1(a) with the dashed line. In this limit, the Bloch dispersion can be neglected and the scattering problem reduces to 2D with the coupling constant Eq. (16). In such conditions, Eq. (8) yields the ground-state energy of a 2D Bose gas in the presence of disorder

$$\frac{E_{g2D}}{L^2} \simeq \frac{1}{2}g_{2D}n_{2D}^2\left[1 - \frac{\ln\left[\ln\left(1/n_{2D}a_{2D}^2\right)\right]}{\ln\left(1/n_{2D}a_{2D}^2\right)} + \frac{B}{\ln\left(1/n_{2D}a_{2D}^2\right)} + \left(\gamma_{2D} + 2R_{2D}\frac{\operatorname{arccoth}\left(\sqrt{1+\frac{2t}{n_{2D}g_{2D}}}\right)}{\ln\left(1/n_{2D}a_{2D}^2\right)}\right)\right], \tag{18}$$

where L^2 is the surface area of the gas, $n_{2D} = n_0 d$ is the surface density, and $B = 1/2 - \ln(mt/n_{2D}2\pi\hbar^2)$. In addition, the two parameters of disorder, respectively, take their 2D value γ_{2D} and R_{2D}. Both parameters, however, depend on the 2D expression of $\tilde{g}_{\mathrm{imp}}$, which needs to be obtained from investigating in detail the 2D scattering problem of a boson with a quenched impurity. Such a problem is definitely nontrivial and shall be left for further investigation. In spite of this, Eq. (18) has shed light on the ground-state properties of a 2D Bose gas in the presence of weak disorder.

Particularly, Eq. (18) presents one of the key results of this paper as follows. First, Eq. (18) in the absence of disorder formally reproduces corresponding results in Ref. [31] for the ground-state energy of a purely 2D dilute Bose gas. From this viewpoint, we expect that the character of a 1D-lattice-confined Bose gas in the presence of weak disorder in the 2D regime will be similar to a purely 2D Bose gas in a random potential. Therefore, we argue that Eq. (18) provides an analytical expression for the ground-state energy of a uniform 2D Bose gas in the presence of weak disorder. Specifically, the last two terms provide the contribution of disorder to the ground-state energy. Second, Eq. (18) has provided beyond-mean-field corrections due to quantum fluctuations in the 2D geometry. These corrections arise from the combined effects of interatomic interaction and disorder, and exhibit in 2D a characteristic $1/\ln(1/n_{2D}a_{2D}^2)$ dependence, in contrast to the 3D counterpart $\sqrt{n_0 a_{3D}^3}$.

In a similar fashion, we analyze the asymptotic behavior of quantum depletion. In the limit $2t/\tilde{g}_e n_0 \gg 1$, corresponding to the anisotropic 3D regime $H(x) \simeq 2/(3\sqrt{x})$. Consequently, one finds the quantum depletion in 3D

$$\left.\frac{\Delta N}{N}\right|_{3D} \simeq \left(\frac{8}{3}+\frac{\pi}{2}\tilde{R}_{3D}\right)\sqrt{\frac{m^*}{m}}\left(\frac{n_0\tilde{a}_{3D}^3}{\pi}\right)^{1/2}, \quad (19)$$

characterized by the dependence on the 3D gas parameter $(n_0\tilde{a}_{3D}^3)^{1/2}$. In the opposite 2D limit, on the other hand, $\tilde{g}_e = g_{2D}d$ and $H(x)$ saturates to the value $\pi/2$. Equation (12) thereby asymptotically approaches the 2D quantum depletion as

$$\left.\frac{\Delta N}{N}\right|_{2D} \simeq \left(1+\frac{R_{2D}}{4}\right)\frac{1}{\ln\left(1/n_{2D}a_{2D}^2\right)}, \quad (20)$$

which is proportional to $1/\ln(1/n_{2D}a_{2D}^2)$, the small parameter in 2D. For varnishing disorder, Eq. (20) is in good agreement with Refs. [20,31] on the quantum depletion of a purely weakly interacting 2D Bose gas. The second term in Eq. (20), therefore, presents the disorder-induced condensate depletion in 2D. Furthermore, a comparison of Eq. (20) with Eq. (19) shows that, in the region where the Bogoliubov theory applies, for the same value of the gas parameter the quantum depletion due to disorder is larger in 2D than in 3D. A similar conclusion has been drawn in Ref. [31] for the quantum depletion induced by interatomic interaction.

IV. SUPERFLUID DENSITY

In this section, we calculate the superfluid density of a dilute Bose gas in the presence of a 1D optical lattice and weak disorder. The general definition of the superfluid density is proposed by Hohenberg and Martin [6]. We emphasize that superfluidity is a kinetic property of a system and superfluid density is essentially a transport coefficient, in contrast to the condensate density which is an equilibrium quantity. Superfluid density can be determined by the response of the system to an external perturbation [6].

In this paper, we adopt the following definition: Supposing that a linear phase $\mathbf{Qr}$ is imposed on the originally static bosonic field which gives rise to a superfluid velocity $\upsilon = \hbar\mathbf{Q}/m$; in response, the thermodynamic potential of the system is changed by [35–37]

$$\frac{\delta\Omega}{V} = \frac{\hbar^2}{2m}\sum_{\alpha\beta} n_{\alpha\beta}Q_\alpha Q_\beta. \quad (21)$$

Here the transport coefficient $n_{\alpha\beta}$ is interpreted as the superfluid density [37]. In general, the $n_{\alpha\beta}$ is a tensor for an anisotropic system.

To obtain $n_{\alpha\beta}$, we substitute the wave function for a flowing condensate $\psi(\mathbf{r},\tau) = \varphi(\mathbf{r},\tau)e^{i\mathbf{Qr}}$ into Eq. (2) and obtain the action $S_{\mathbf{Q}}$ for the superfluid

$$S_{\mathbf{Q}} = S + \hbar\beta V\sum_{\mathbf{k},n}\psi_{\mathbf{k},n}^*\left[f_{\mathbf{kQ}}+\frac{\hbar^2}{2m}Q^2\right]\psi_{\mathbf{k},n}, \quad (22)$$

where S refers to the action for a static BEC in Eq. (2) and $f_{\mathbf{kQ}} = [\hbar^2/m(k_xQ_x+k_yQ_y)+2Q_ztd\sin(k_zd)]$. Proceeding in a similar fashion as in Sec. II, we obtain

$$\begin{aligned}\Omega_{\mathbf{Q}} = V&\left(-\tilde{\mu}n_0 + n_{\text{imp}}\tilde{g}_{\text{imp}}n_0 + \frac{\tilde{g}_e n_0^2}{2}\right)\\ &-\frac{1}{2}\sum_{\mathbf{k}\neq\mathbf{0}}\left(\varepsilon_{\mathbf{k}}^0 - \tilde{\mu} + 2\tilde{g}_e n_0 - \widetilde{E}_{\mathbf{k}}\right)\\ &- n_{\text{imp}}\tilde{g}_{\text{imp}}^2 n_0\sum_{\mathbf{k}\neq\mathbf{0}}\frac{\varepsilon_{\mathbf{k}} - \tilde{\mu} + \tilde{g}_e n_0}{\widetilde{E}_{\mathbf{k}}^2 - f_{\mathbf{kQ}}^2},\end{aligned} \quad (23)$$

where $\widetilde{E}_{\mathbf{k}} = \sqrt{(\varepsilon_{\mathbf{k}}^0 - \widetilde{\mu} + 2\tilde{g}_e n_0)^2 - \tilde{g}_e^2 n_0^2}$ depends on $\mathbf{Q}$ though $\tilde{\mu} = \mu - \hbar^2\mathbf{Q}^2/2m$.

Since the presence of a 1D optical lattice breaks the global rotational symmetry and leaves the gas system only isotropic in the x-y plane, one can write $n_{\alpha\beta} = n_{\alpha\alpha}\delta_{\alpha\beta}$ where $n_{xx} = n_{yy} \neq n_{zz}$. Expanding Eq. (23) in powers of $\mathbf{Q}$ and truncating at the quadratic order, we compare the resulting expression with Eq. (21) and obtain

$$n_{xx} = n_{yy} = n - \frac{2n_{\text{imp}}\tilde{g}_{\text{imp}}^2 n_0}{V}\sum_{\mathbf{k}\neq\mathbf{0}}\frac{\hbar^2k_x^2}{m}\frac{\varepsilon_{\mathbf{k}}^0}{E_{\mathbf{k}}^4}, \quad (24)$$

and

$$n_{zz} = n - \frac{2mn_{\text{imp}}\tilde{g}_{\text{imp}}^2 n_0}{\hbar^2 V}\sum_{\mathbf{k}\neq\mathbf{0}}\frac{\varepsilon_{\mathbf{k}}^0}{E_{\mathbf{k}}^4}\left[2td\sin(k_zd)\right]^2. \quad (25)$$

Similar results have been obtained in Ref. [22] using the current-current response function. The formal agreement between the two affirms that, in spite of different ways to impose perturbation and various options of physical quantities to measure the response, these different routes to obtain superfluid density can be unified within the framework of the linear response theory.

The disorder-induced normal fluid density fraction can be obtained through $(n_n)_{\alpha\beta} = (1 - n_{\alpha\alpha}/n)\delta_{\alpha\beta}$. Taking the continuum limit of Eqs. (24) and (25), one finds

$$(n_n)_{xx} = (n_n)_{yy} = \tilde{R}\frac{m\tilde{g}_e}{8\hbar^2\pi d}I\left(\frac{2t}{\tilde{g}_e n_0}\right), \quad (26)$$

and

$$(n_n)_{zz} = \tilde{R}\left(\frac{m}{m^*}\right)^2\frac{1}{16\pi n_0 d^3}K\left(\frac{2t}{\tilde{g}_e n_0}\right), \quad (27)$$

where $I(x)$ and $K(x)$ are functions of the variable $x = 2t/\tilde{g}_e n_0$, respectively, defined as

$$I(x) = \left[\sqrt{1+x} - x\ln\left(\frac{1+\sqrt{1+x}}{\sqrt{x}}\right)\right], \quad (28)$$

and

$$K(x) = \ln\left(\frac{1+\sqrt{1+x}}{\sqrt{x}}\right) - \frac{2-(2-x)\sqrt{1+x}}{x^2}. \quad (29)$$

The results of Eqs. (28) and (29) are plotted in Fig. 2. In the asymptotic 3D limit, one finds $I(x) \simeq 2/3\sqrt{x}$ and $K(x) \simeq 4/3x^{3/2}$, corresponding to the dashed curves in Fig. 2. In such

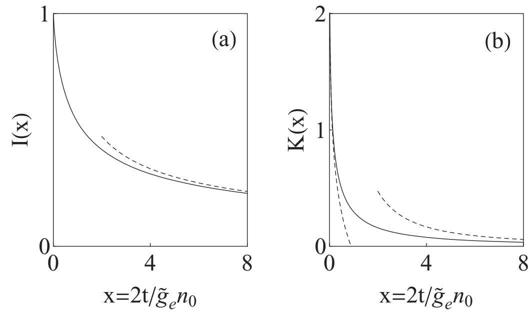

FIG. 2. (a) Scaling function $I(x)$ in Eq. (28) (solid line) and its asymptotic behavior (dashed line). (b) Scaling function $K(x)$ in Eq. (29) (solid line) and its asymptotic behavior (dashed line).

a situation, Eqs. (26) and (27), respectively, become

$$(n_n)_{xx} = (n_n)_{yy} \simeq \frac{2\pi}{3}\tilde{R}_{3D}\sqrt{\frac{m^*}{m}}\left(\frac{n_0\tilde{a}_{3D}^3}{\pi}\right)^{\frac{1}{2}}, \qquad (30)$$

and

$$(n_n)_{zz} \simeq \frac{2\pi}{3}\tilde{R}_{3D}\sqrt{\frac{m}{m^*}}\left(\frac{n_0\tilde{a}_{3D}^3}{\pi}\right)^{\frac{1}{2}}. \qquad (31)$$

Equation (31) demonstrates a similar dependence on the 3D gas parameter as the 3D quantum depletion in Eq. (19). Moreover, the ratio between Eq. (30) and the disorder-induced quantum depletion n_d in Eq. (19) equals 4/3 in the unconfined $X(Y)$ direction, in agreement with Ref. [21], whereas this ratio becomes $(n_n)_{zz}/n_d = 4m^*/3m$ due to the increased inertia of the gas along the direction of optical lattice [22].

In the opposite 2D limit, one obtains the limiting expression $I(x) \simeq 1$ and $(n_n)_{2D} = (n_n)_{xx} = (n_n)_{yy}$ is found to be

$$(n_n)_{2D} \simeq \frac{R_{2D}}{2}\frac{1}{\ln\left(1/n_{2D}a_{2D}^2\right)}. \qquad (32)$$

Equation (32) presents another key result of this paper, providing an analytical expression for the normal fluid density in a homogenous Bose fluid in 2D in the presence of weak disorder. Equation (32) shows that the normal fluid density in 2D exhibits a characteristic $1/\ln(1/n_{2D}a_{2D}^2)$ dependence. With respect to the 3D case, a comparison of Eqs. (20) and (32) leads to $n_n/n_d = 2$ in 2D, indicating a more pronounced contrast between superfluidity and BEC at $T = 0$. On the other hand, $K(x)$ in Eq. (29) diverges in the limit $x \to 0$, leading to diverging n_{zz} in Eq. (27) for vanishing tunneling. This signals the absence of superfluidity along the direction of the optical lattice, which is consistent with the kinematical 2D nature of the Bose gas in the absence of tunneling along the direction of the laser.

V. POSSIBLE EXPERIMENTAL SCENARIOS AND CONCLUSION

Central to testing the validity of the physics in this article concerns the experimental realization of a BEC in the superfluid phase along the entire evolution from 3D to quasi-2D. The present facilities have allowed one to adjust the depth of a lattice, realize tight confinement of the motion of trapped particles, and ultimately achieve a kinematically 2D gas. In typical experiments to date, quasi-2D quantum degenerate Bose gases have been experimentally produced both in single pancake traps and at the nodes of 1D optical lattice potentials [15]. In addition, it has been suggested that BEC and superfluidity can both be achieved below a critical temperature [14]. Furthermore, adding a tunable periodic potential allows one to combine the benefit of the reduced dimensionality with the advantage of working with large, yet coherent samples [27].

Upon overcoming the previous difficulties, the experimental realization of our scenario amounts to controlling three parameters whose interplay underlies the physics of this work: the strength of an optical lattice s, the interaction between bosonic atoms $\tilde{g}n_0$, and the strength of disorder $\tilde{R}$. All these quantities are experimentally controllable using state-of-the-art technologies. The interatomic interaction can be controlled in a very versatile manner via the technology of Feshbach resonances [38]. In the typical experiments to date, the values of ratio $\tilde{g}n_0/E_R$ range from 0.02 to 1 [39,40]. The depth of an optical lattice s can be changed from $0E_R$ to $32E_R$ almost at will [41]. Disorder may be created in a repeatable way by introducing impurities in the sample [42], or using laser speckles and multichromatic lattices [43–45].

Further difficulties may arise in measuring the beyond-mean-field corrections to the ground-state energy along the dimensional crossover. For typical values of the atom density and scattering length, such corrections remain very small and hard to observe in the usual experiments that measure density profiles or release energy. They can be visible, however, in the frequencies of collective excitations in a lattice system [27,46,47]. The direct measurement of quantum depletions of a quasi-2D condensate can be achieved either through observing ballistic expansion [48] or applying Bragg spectroscopy [49]. It is worth mentioning that the possibility to use ballistic expansion to measure quantum fluctuations is associated with the characteristics of an optical lattice where the confinement frequency at each lattice site far exceeds the interaction energy. As such, the time-of-flight images are essentially a snapshot of the frozen-in momentum distribution of the wave function at the time of the lattice switch-off, thus allowing for a direct observation of quantum depletions. This technology cannot be applied, for example, to measure quantum depletions of a quasi-2D Bose gas confined in a harmonic trap. From this perspective, Bragg spectroscopy admits broader ranges of application, independent of methods of confinement to create quasi-2D BEC's systems.

We expect, therefore, that the phenomena discussed in this article should be observable within the current experimental capability. We emphasize here that the presented work is restricted to weak disorder and weak interatomic interaction. For further investigations in the presence of stronger interatomic interaction or disorder, the path-integral Monte Carlo simulation is a reliable method [50].

In summary, we have investigated a dilute Bose gas trapped in a 1D optical lattice and a random potential. Capitalizing on the characteristic dimensional crossover properties, the obtained results in the quasi-2D regime allow us to derive analytical expressions for the ground-state energy, quantum depletion, and superfluid density of an effectively pure 2D

Bose gas in the presence of weak disorder. Our analysis signifies a more pronounced effect of disorder in systems with reduced dimensionality in enhancing quantum fluctuations and depleting superfluid density. In particular, the ratio between the normal fluid density and the corresponding condensate depletion increases to 2 in 2D, in contrast to the familiar 4/3 in lattice-free 3D geometry.

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[1] D. S. Petrov, D. M. Gangardt, and G. V. Shlyapnikov, J. Phys. IV France **116**, 5 (2004).
[2] A. Posazhennikova, Rev. Mod. Phys. **78**, 1111 (2006).
[3] L. P. Pitaevskii and S. Stringari, *Bose-Einstein Condensation* (Clarendon Press, Oxford, 2003).
[4] N. N. Bogoliubov, *Selected Works, Part II: Quantum and Statistical Mechanics* (Gordon and Breach, New York, 1991).
[5] N. D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966).
[6] P. C. Hohenberg, Phys. Rev. **158**, 383 (1967).
[7] W. Kane and L. Kadanoff, Phys. Rev. **155**, 80 (1967).
[8] V. L. Berezinskii, Sov. Phys. JETP **32**, 493 (1971); **34**, 610 (1972).
[9] J. M. Kosterlitz and D. J. Thouless, J. Phys. C **6**, 1181 (1973); J. M. Kosterlitz, *ibid.* **7**, 1046 (1974).
[10] V. N. Popov, *Functional Integrals in Quantum Field Theory and Statistical Physics* (Reidel, Dordrecht, 1983).
[11] C. Mora and Y. Castin, Phys. Rev. A **67**, 053615 (2003); Phys. Rev. Lett. **102**, 180404 (2009).
[12] I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. **80**, 885 (2008).
[13] U. Al Khawaja, J. O. Andersen, N. P. Proukakis, and H. T. C. Stoof, Phys. Rev. A **66**, 013615 (2002); J. O. Andersen, U. Al Khawaja, and H. T. C. Stoof, Phys. Rev. Lett. **88**, 070407 (2002).
[14] D. S. Petrov, M. Holzmann, and G. V. Shlyapnikov, Phys. Rev. Lett. **84**, 2551 (2000); D. S. Petrov and G. V. Shlyapnikov, Phys. Rev. A **64**, 012706 (2001).
[15] A. Görlitz *et al.*, Phys. Rev. Lett. **87**, 130402 (2001); D. Rychtarik, B. Engeser, H.-C. Nägerl, and R. Grimm, *ibid.* **92**, 173003 (2004); S. Burger *et al.*, Europhys. Lett. **57**, 1 (2002); M. Köhl *et al.*, J. Low Temp. Phys. **138**, 635 (2005); P. Cladé *et al.*, Phys. Rev. Lett. **102**, 170401 (2009).
[16] U. R. Fischer, Phys. Rev. Lett. **89**, 280402 (2002); J. Low Temp. Phys. **138**, 723 (2005).
[17] V. L. Yukalov, Laser Phys. **19**, 1 (2009).
[18] L. Sanchez-Palencia and M. Lewenstein, Nature Phys. **6**, 87 (2010).
[19] B. Deissler, M. Zaccanti, G. Roati, C. D'Errico, M. Fattori, M. Modugno, G. Modugno, and M. Inguscio, Nature Phys. **6**, 354 (2010).
[20] M. Schick, Phys. Rev. A **3**, 1067 (1971).
[21] J. O. Andersen, Eur. J. Phys B **28**, 389 (2002).
[22] K. Huang and H. F. Meng, Phys. Rev. Lett. **69**, 644 (1992); A. V. Lopatin and V. M. Vinokur, *ibid.* **88**, 235503 (2002).
[23] Y. Hu, Z. X. Liang, and B. B. Hu, Phys. Rev. A **80**, 043629 (2009); **81**, 053621 (2010).
[24] G. E. Astrakharchik, J. Boronat, J. Casulleras, and S. Giorgini, Phys. Rev. A **66**, 023603 (2002).
[25] P. O. Fedichev, M. J. Bijlsma, and P. Zoller, Phys. Rev. Lett. **92**, 080401 (2004).
[26] M. Wouters and G. Orso, Phys. Rev. A **73**, 012707 (2006).
[27] G. Orso, C. Menotti, and S. Stringari, Phys. Rev. Lett. **97**, 190408 (2006).
[28] G. M. Falco, A. Pelster, and R. Graham, Phys. Rev. A **75**, 063619 (2007).
[29] N. M. Hugenholtz and D. Pines, Phys. Rev. **116**, 489 (1959).
[30] E. M. Lifshitz and L. P. Pitaevskii, *Statistical Physics, Part II* (Pergamon, Oxford, 1980).
[31] S. Pilati, J. Boronat, J. Casulleras, and S. Giorgini, Phys. Rev. A **71**, 023605 (2005), and references therein.
[32] M. Krämer, L. Pitaevskii, and S. Stringari, Phys. Rev. Lett. **88**, 180404 (2002).
[33] Z. X. Liang, X. Dong, Z. D. Zhang, and B. Wu, Phys. Rev. A **78**, 023622 (2008).
[34] T. D. Lee and C. N. Yang, Phys. Rev. **105**, 1119 (1957); T. D. Lee, K. W. Huang, and C. N. Yang, *ibid.* **106**, 1135 (1957).
[35] A. M. Rey, K. Burnett, R. Roth, M. Edwards, C. J. Williams, and C. W. Clark, J. Phys. B: At. Mol. Opt. Phys. **36**, 825 (2003).
[36] T. Paananen, J. Phys. B: At. Mol. Opt. Phys. **42**, 165304 (2009).
[37] A. J. Legget, Rev. Mod. Phys. **71**, S318 (1999).
[38] E. Tiesinga, B. J. Verhaar, and H. T. C. Stoof, Phys. Rev. A **47**, 4114 (1993); S. Inouye *et al.*, Nature (London) **392**, 151 (1998).
[39] O. Morsch and M. Oberthaler, Rev. Mod. Phys. **78**, 179 (2006).
[40] I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. **80**, 885 (2008).
[41] M. Greiner, O. Mandel, T. Esslinger, T. W. Hansch, and J. Bloch, Nature (London) **415**, 39 (2002).
[42] S. Ospelkaus, C. Ospelkaus, O. Wille, M. Succo, P. Ernst, K. Sengstock, and K. Bongs, Phys. Rev. Lett. **96**, 180403 (2006).
[43] R. Roth and K. Burnett, Phys. Rev. A **68**, 023604 (2003).
[44] B. Damski, J. Zakrzewski, L. Santos, P. Zoller, and M. Lewenstein, Phys. Rev. Lett. **91**, 080403 (2003).
[45] U. Gavish and Y. Castin, Phys. Rev. Lett. **95**, 020401 (2005).
[46] L. Pitaevskii and S. Stringari, Phys. Rev. Lett. **81**, 4541 (1998).
[47] A. Altmeyer, S. Riedl, C. Kohstall, M. J. Wright, R. Geursen, M. Bartenstein, C. Chin, J. H. Denschlag, and R. Grimm, Phys. Rev. Lett. **98**, 040401 (2007).
[48] K. Xu, Y. Liu, D. E. Miller, J. K. Chin, W. Setiawan, and W. Ketterle, Phys. Rev. Lett. **96**, 180405 (2006).
[49] J. M. Vogels, K. Xu, C. Raman, J. R. Abo-Shaeer, and W. Ketterle, Phys. Rev. Lett. **88**, 060402 (2002); F. Zambelli, L. Pitaevskii, D. M. Stamper-Kurn, and S. Stringari, Phys. Rev. A **61**, 063608 (2000).
[50] S. Pilati, S. Giorgini, M. Modugno, and N. Prokof'ev, New J. Phys. **12**, 073003 (2010).