Fidelity susceptibility and topological phase transition of a two-dimensional spin-orbit-coupled Fermi superfluid

Xuebing Luo,[1] Kezhao Zhou,[1,2,*] Wuming Liu,[3] Zhaoxin Liang,[1] and Zhidong Zhang[1]
[1]*Shenyang National Laboratory for Materials Science, Institute of Metal Research, Chinese Academy of Sciences, 72 Wenhua Road, Shenyang 110016, China*
[2]*Department of Physics, University of Illinois, 1110 W. Green Street, Urbana, IL 61801*
[3]*Institute of Physics, Chinese Academy of Sciences, Beijing 100080, China*
(Received 5 November 2013; revised manuscript received 28 December 2013; published 21 April 2014)

We investigate the fidelity susceptibility (FS) of a two-dimensional spin-orbit-coupled (SOC) Fermi superfluid and the topological phase transition driven by a Zeeman field in the perspective of its ground state wave function. Without Zeeman coupling, FS shows additional features characterizing the BCS-BEC crossover induced by SOC. In the presence of a Zeeman field, the topological phase transition is explored using both FS and the topological invariant. In particular, we obtain the analytical result of the topological invariant which explicitly demonstrates that the topological phase transition corresponds to a sudden change of the ground state wave function. Consequently, FS diverges at the phase transition point with its critical behavior being $\chi \propto \ln|h - h_c|$. Based on this observation, we conclude that the topological phase transition can be detected by measuring the momentum distribution in cold atom experiments.

DOI: 10.1103/PhysRevA.89.043612 PACS number(s): 67.85.Lm, 05.30.Fk, 71.10.Pm

I. INTRODUCTION

Spin-orbit coupling (SOC) is a key ingredient in realizing nontrivial topological phases which is of great interest in the current physics community [1–3]. Particularly, semiconductors with proximity-induced superconductivity and external effective Zeeman coupling as well as noncentrosymmetric superconductors provide real material examples with a non-Abelian quantum order [4]. Another promising platform of achieving this scenario is ultracold Fermi systems where SOC, Zeeman coupling, and superconductivity can be readily realized in the current experimental setup [5–7] and construction of model systems on the Hamiltonian level is now available [8,9]. Furthermore, two-dimensional (2D) geometry can be created by generating a strong trapping potential along one direction [10–12].

Theoretical investigations have shown that SOC has nontrivial effects on various properties of the Fermi superfluid systems. In the absence of Zeeman coupling, SOC can produce a bound state called Rashbons [13,14] and therefore induce a crossover from weakly correlated BCS to a strongly interacting BEC regime (BCS-BEC) even for very weak particle-particle interactions [15–18]. Of particular interest is the topological phase transition driven by a Zeeman field [19–22] which can be classified by a topological invariant constructed for this particular scenario [4] and also in He^3-A [23], where its theoretical structure and relation with other invariants have been discussed comprehensively. However, much of the information encoded in the topological invariant has been buried in the numerical procedure and therefore an analytical result is of great interest and value for the full understanding of these states. Both of these aspects are fundamentally related to the ground state wave function of the many-body systems considered. As a direct measure of the change of the ground state wave function, fidelity susceptibility (FS) has been used extensively to study the quantum phase transition problems [24], which is the main motivation of this paper.

In this paper we consider a typical 2D s-wave Fermi superfluid in the presence of both SOC and Zeeman coupling. In the absence of Zeeman coupling, with the self-consistent solution of the gap and number equations, we investigate the behaviors of fidelity susceptibility (FS) as functions of both interaction and SOC. Numerical results show that FS exhibits a local peak structure characterizing the BCS-BEC crossover induced by SOC which is quite different from other thermodynamic quantities. In the presence of Zeeman coupling, we focus on a situation with a strong enough SOC such that only topological phase transition occurs. In particular, we obtain the analytical result of the topological invariant which provides additional insights into the topological nature of the ground state: (i) Singular behavior of the Berry curvature is determined by SOC through the vortexlike solutions of the Bogoliubov–de Gennes (BdG) Hamiltonian instead of the Dirac point in the excitation spectrum acting as a magnetic monopole in momentum space; and (ii) the topological phase transition corresponds to a sudden change of the ground state wave function at zero momentum which is also captured by the divergence of FS. Finally, its critical behavior is obtained by analyzing the gapless excitation spectrum at the phase transition point.

II. FORMALISM

The system under consideration can be described by the Hamiltonian $H = H_0 - g\int d^2\mathbf{r}\varphi_\uparrow^\dagger(\mathbf{r})\varphi_\downarrow^\dagger(\mathbf{r})\varphi_\downarrow(\mathbf{r})\varphi_\uparrow(\mathbf{r})$, with $g > 0$ being the contact interaction parameter and $\varphi_{\sigma(=\uparrow,\downarrow)}(\mathbf{r})$ and $\varphi_\sigma^\dagger(\mathbf{r})$ are the annihilation and creation field operators, respectively. The noninteracting part H_0 can be written as $H_0 = \int d^2\mathbf{r}\psi^\dagger(\mathbf{r})[\varepsilon_{\hat{\mathbf{p}}} - h\sigma_z + \lambda(\sigma \times \hat{\mathbf{p}})\cdot\mathbf{z}]\psi(\mathbf{r})$, where $\psi(\mathbf{r}) = [\varphi_\uparrow(\mathbf{r}),\varphi_\downarrow(\mathbf{r})]^T$ and kinetic energy $\varepsilon_{\hat{\mathbf{p}}} = \hat{\mathbf{p}}^2/2m - \mu$, with m, μ, and h being the mass of the Fermi atoms, the chemical potential, and the effective Zeeman field, respectively.

*kezhaozhou@gmail.com

For simplicity we set $\hbar = 1$ throughout this paper. The third term is the Rashba SOC [25] with $\lambda > 0$ denoting the strength of SOC and σ being the Pauli matrices. Within mean-field theory, the interacting part can be approximated by $H_{\rm BCS} = -\int d^2\mathbf{r}[\Delta(\mathbf{r})\varphi_\uparrow^\dagger(\mathbf{r})\varphi_\downarrow^\dagger(\mathbf{r}) + \text{H.c.}] + \int d^2\mathbf{r}|\Delta(\mathbf{r})|^2/g$, with $\Delta(\mathbf{r})$ being the pairing field.

This model Hamiltonian $H = H_0 + H_{\rm BCS}$ is also relevant to semiconductors where superconductivity is induced by proximity effect and noncentrosymmetric superconductors in the sense that the p-wave pairing component does not affect much the topological properties of the system. As shown in Ref. [4], the Majorana zero-energy state for the vortex solution of the BdG equation contains the essential details of the nontrivial topological nature of the ground state. For our purpose, we only consider translational invariant solutions where the paring field becomes a constant $\Delta(\mathbf{r}) = \Delta$. Consequently, in momentum space, the total Hamiltonian reduces to $H = \sum_{\mathbf{p}>0}\Phi_\mathbf{p}^\dagger H_{\rm BdG}(\mathbf{p})\Phi_\mathbf{p} + \sum_\mathbf{p}\varepsilon_\mathbf{p} + V\Delta^2/g$, where V denotes the size of the system, $\Phi_\mathbf{p} = [a_{\mathbf{p},\uparrow}, a_{\mathbf{p},\downarrow}, a_{-\mathbf{p},\uparrow}^\dagger, a_{-\mathbf{p},\downarrow}^\dagger]^T$, and the BdG Hamiltonian $H_{\rm BdG}(\mathbf{p})$ is

$$H_{\rm BdG}(\mathbf{p}) = \begin{bmatrix} \varepsilon_\mathbf{p} - h & \Gamma_\mathbf{p} & 0 & -\Delta \\ \Gamma_\mathbf{p}^* & \varepsilon_\mathbf{p} + h & \Delta & 0 \\ 0 & \Delta & -\varepsilon_\mathbf{p} + h & \Gamma_\mathbf{p}^* \\ -\Delta & 0 & \Gamma_\mathbf{p} & -\varepsilon_\mathbf{p} - h \end{bmatrix}, \quad (1)$$

with $\Gamma_\mathbf{p} = \lambda(p_y + ip_x)$. Using the standard diagonalization procedure, we obtain the ground-state free energy $E_g = \sum_{\mathbf{p},s=\pm}(\varepsilon_\mathbf{p} - E_{\mathbf{p},s})/2 + V\Delta^2/g$, where the excitation spectrum $E_{\mathbf{p},s} = \sqrt{\mathcal{E}_{\mathbf{p},s}^2 + \Delta_{\mathbf{p},2}^2}$, with $\mathcal{E}_{\mathbf{p},s} = E_\mathbf{p} - s\sqrt{h^2 + |\Gamma_\mathbf{p}|^2}$, $E_\mathbf{p} = \sqrt{\varepsilon_\mathbf{p}^2 + \Delta_{\mathbf{p},1}^2}$, $\Delta_{\mathbf{p},1} = \Delta|\cos\theta_\mathbf{p}|$, $\Delta_{\mathbf{p},2} = \Delta\sin\theta_\mathbf{p}$, and $\theta_\mathbf{p} = \pi - \arctan(|\Gamma_\mathbf{p}|/h)$. For fixed h, λ, and g, Δ and μ are given by the self-consistent solutions of the gap and number equations:

$$\frac{1}{g} = \frac{1}{V}\sum_{\mathbf{p},s}\frac{1 + s\cos\theta_\mathbf{p}\frac{h}{E_\mathbf{p}}}{4E_{\mathbf{p},s}}, \quad (2)$$

$$N = \frac{1}{2}\sum_{\mathbf{p},s}\left(1 - \frac{\mathcal{E}_{\mathbf{p},s}}{E_{\mathbf{p},s}}\frac{\varepsilon_\mathbf{p}}{E_\mathbf{p}}\right). \quad (3)$$

Note that in the presence of Zeeman coupling, Eq. (2) generally supports more than one solution, while the physical one corresponds to the global minimum point of E_g. As usual, divergence of the integral over momenta in Eq. (2) is removed by replacing contact interaction parameter g by binding energy E_b through $V/g = \sum_\mathbf{p} 1/(2\epsilon_\mathbf{p} + E_b)$.

Furthermore, the ground state wave function can be directly given by the unitary transformation that diagonalizes Eq. (1). However, we find that the final result becomes more physically transparent by using Bogoliubov transformation step by step. First, in helicity basis: $c_{\mathbf{p},s} = \sin(\theta_\mathbf{p}/2)a_{\mathbf{p},s} - s\cos(\theta_\mathbf{p}/2)e^{is\varphi_\mathbf{p}}a_{\mathbf{p},-s}$, with $\varphi_\mathbf{p} = \arctan(p_x/p_y)$, the total Hamiltonian becomes $H = V\Delta^2/g + \sum_{\mathbf{p},s}\xi_{\mathbf{p},s}c_{\mathbf{p},s}^\dagger c_{\mathbf{p},s} - 1/2\sum_{\mathbf{p},s}(\Delta_{\mathbf{p},2}e^{is\varphi_\mathbf{p}}c_{\mathbf{p},s}^\dagger c_{-\mathbf{p},s}^\dagger - s\Delta_{\mathbf{p},1}c_{\mathbf{p},s}^\dagger c_{-\mathbf{p},-s}^\dagger + \text{H.c.})$, with $\xi_{\mathbf{p},s} = \varepsilon_\mathbf{p} + s|\Gamma_\mathbf{p}|$ from which we see that pairing happens between both the same and different helicity bases. Second, using Bogoliubov transformation: $\beta_{\mathbf{p},s} = u_\mathbf{p}c_{\mathbf{p},s} - v_\mathbf{p}c_{-\mathbf{p},-s}^\dagger$, with $u_\mathbf{p} = \sqrt{(1 + \varepsilon_\mathbf{p}/E_\mathbf{p})/2}$ and $u_\mathbf{p}^2 + v_\mathbf{p}^2 = 1$, the Hamiltonian reduces to the standard pairing form as $H = V\Delta^2/g + \sum_\mathbf{p}(\varepsilon_\mathbf{p} - E_\mathbf{p}) + \sum_{\mathbf{p},s}\mathcal{E}_{\mathbf{p},s}\beta_{\mathbf{p},s}^\dagger\beta_{\mathbf{p},s} - 1/2\sum_{\mathbf{p},s}\Delta_{\mathbf{p},2}(e^{is\varphi_\mathbf{p}}\beta_{\mathbf{p},s}^\dagger\beta_{-\mathbf{p},s}^\dagger + \text{H.c.})$. Finally, the ground state wave function can be easily obtained as

$$|G\rangle = \prod_{\mathbf{p}>0,s}(u_{\mathbf{p},s} + e^{is\varphi_\mathbf{p}}v_{\mathbf{p},s}\beta_{\mathbf{p},s}^\dagger\beta_{-\mathbf{p},s}^\dagger)|g\rangle, \quad (4)$$

where $|g\rangle = \prod_\mathbf{p}(u_\mathbf{p} + v_\mathbf{p}c_{\mathbf{p},+}^\dagger c_{-\mathbf{p},-}^\dagger)|0\rangle$, $u_{\mathbf{p},s}$ and $v_{\mathbf{p},s}$ are given as

$$\begin{bmatrix} u_{\mathbf{p},s} \\ v_{\mathbf{p},s} \end{bmatrix} = \sqrt{\frac{1}{2}\left(1 \pm \frac{\mathcal{E}_{\mathbf{p},s}}{E_{\mathbf{p},s}}\right)}. \quad (5)$$

Here $|g\rangle$, constructed to be the vacuum state of $\beta_{\mathbf{p},s}$, can be considered as singlet pairing of different helicity states ($c_{\mathbf{p},s}$). The ground state wave function $|G\rangle$ describes triplet pairing of the quasiparticles denoted by $\beta_{\mathbf{p},s}$. In the absence of Zeeman coupling, $u_\mathbf{p} = 1$, $v_\mathbf{p} = 0$, and $|g\rangle = |0\rangle$, $\beta_{\mathbf{p},s}$ is the helicity basis and $|G\rangle$ represents state with pairing only happening in the same helicity basis [15].

Finally, as a direct measure of the change of ground state wave function, FS is defined as the following form:

$$\chi(\alpha) = \langle G|\overleftarrow{\partial}_\alpha\partial_\alpha|G\rangle - \langle G|\overleftarrow{\partial}_\alpha|G\rangle\langle G|\partial_\alpha|G\rangle, \quad (6)$$

where $|G\rangle$ denotes the ground state wave function of the many-body systems and α is the control parameter. Recent investigations show that it provides an effective way of determining the phase transition boundary [24]. The critical behaviors of FS near quantum phase transition are of great interest especially for topological phase transitions. On the other hand, it has also been used to study the crossover induced by interaction in the absence of SOC with its width being associated with the crossover region [26].

III. BALANCED CASE

For balanced case $h = 0$, self-consistent solution of the gap and number equations gives μ/E_F and Δ/E_F as functions of $\tilde\lambda = m\lambda/k_F$ and $\eta = E_B/E_F$, with $E_F = k_F^2/2m$ being the Fermi energy and the k_F being defined through $k_F^2 = 2\pi n$. Based on these results, we investigate the effect of SOC on the behaviors of the FS in the BCS-BEC crossover problem. By direct substitution of Eq. (4) into Eq. (6) we obtain

$$\chi(\lambda) = \sum_{\mathbf{p},s}\frac{1}{8E_{\mathbf{p},s}^4}\left[\Delta\left(sp_\perp - \frac{\partial\mu}{\partial\lambda}\right) - \xi_{\mathbf{p},s}\frac{\partial\Delta}{\partial\lambda}\right]^2, \quad (7)$$

$$\chi(g) = \sum_{\mathbf{p},s}\frac{1}{8E_{\mathbf{p},s}^4}\left[\Delta\frac{\partial\mu}{\partial g} + \xi_{\mathbf{p},s}\frac{\partial\Delta}{\partial g}\right]^2, \quad (8)$$

with $p_\perp = \sqrt{p_x^2 + p_y^2}$.

The numerical results of Eqs. (7) and (8) are presented in Fig. 1, where results in three dimensions (3D) are also included as it provides the essential features of FS in the BCS-BEC crossover problem. In 3D, the interaction is characterized by the scattering length a introduced by substituting

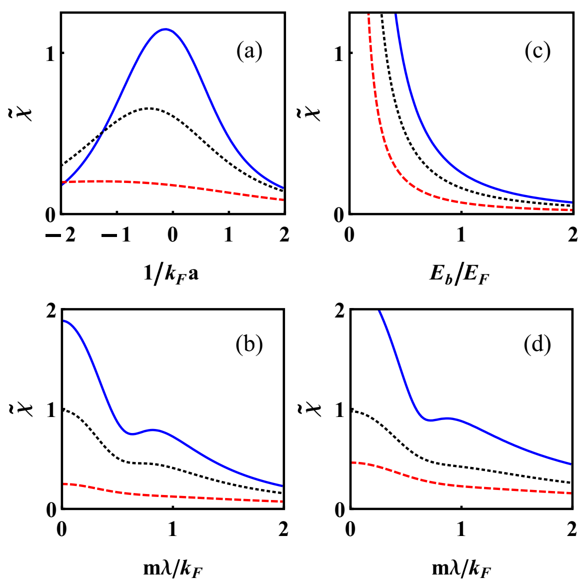

FIG. 1. (Color online) Fidelity susceptibility (in arbitrary dimension) as functions of interaction parameters and strength of SOC in (a) and (b) three dimensions and (c) and (d) two dimensions. In (a) and (c), $\tilde{\lambda}$ are given as 0, 1, and 2 for the solid blue line, black dotted line, and red dashed line, respectively. The interaction parameters are set to be $\eta = 1/(k_F a) = -1.2, -1, -0.6$ in (b) and $\eta = E_b/E_F = 0.2, 0.8, 2.2$ in (d) for lines from above.

$V/g = -mV/4\pi a + \sum_{\mathbf{p}} 1/(2\epsilon_{\mathbf{p}})$ into Eq. (2) and Fermi momentum is defined by $k_F^3 = 3\pi^2 n$.

For 3D case, as can be seen from Fig. 1(a), there is a global peak in FS as a function of interaction parameter $1/k_F a$ and it disappears for strong enough SOC. Besides, it is no longer symmetric for finite strength of SOC. However, Fig. 1(b) shows that around the critical value of λ FS has a local maximum which marks the crossover induced by SOC and also disappears for large enough interaction parameters. For the 2D case, as presented in Figs. 1(c) and 1(d), this feature becomes more distinct in the sense that FS as a function of E_B does not show a clear crossover signature but it does with varying SOC. Finally, it is worthwhile to note that the local peak of FS as a function of λ is not located at the same points where the gap parameter increases suddenly or superfluid density has a minimum value [15,27]. In summary, FS provides a different angle to investigate the effects of SOC in the BCS-BEC crossover problem.

IV. IMBALANCED CASE

Here we only focus on the topological phase transition in 2D. For weak SOC, there will be a first-order phase transition with increasing h and FS is apparently divergent due to the trivial sudden change of Δ and therefore the wave function. For strong enough SOC, Δ is always nonzero even for very large h and decreases continuously with increasing h [22]. However, there is a topological phase transition across $h = h_c = \sqrt{\mu^2 + \Delta^2}$. For $h < h_c$, the system is in a trivial gapped superfluid state. When $h > h_c$, the ground state is nontrivial and is characterized by a nonzero topological invariant $\mathcal{N}$ which is defined as [4,23] $\mathcal{N} = 1/2\pi \int_{-\infty}^{+\infty} d^2\mathbf{p} B(\mathbf{p})$, where the Berry curvature is given by

$$B(\mathbf{p}) = -i \sum_{E_{\mathbf{p}}^{\alpha} < 0} \left[\partial_{p_x} \mathbf{u}_{\mathbf{p},\alpha}^{\dagger} \partial_{p_y} \mathbf{u}_{\mathbf{p},\alpha} - \partial_{p_y} \mathbf{u}_{\mathbf{p},\alpha}^{\dagger} \partial_{p_x} \mathbf{u}_{\mathbf{p},\alpha} \right], \quad (9)$$

with $\mathbf{u}_{\mathbf{p},\alpha=1,2,3,4}$ being the eigenvectors of Eq. (1) corresponding to the eigenvalues $-E_{\mathbf{p},+}, E_{\mathbf{p},+}, -E_{\mathbf{p},-}, E_{\mathbf{p},-}$, respectively. It is clear that the ordering and sign of the elements of the eigenvector do not influence the Berry curvature and therefore $\mathcal{N}$. Furthermore, only negative eigenvectors appears in $\mathbf{B}(\mathbf{p})$. Therefore, we can rewrite the eigenvector in more transparent form as $\mathbf{u}_{\mathbf{p},s=\pm} = [e^{is\varphi_{\mathbf{p}}} F_{\mathbf{p},s}^1, F_{\mathbf{p},s}^2, F_{\mathbf{p},s}^3, e^{is\varphi_{\mathbf{p}}} F_{\mathbf{p},s}^4]^T$, with

$$F_{\mathbf{p},s}^1 = u_{\mathbf{p}} \sin\frac{\theta_{\mathbf{p}}}{2} v_{\mathbf{p},s} - v_{\mathbf{p}} \cos\frac{\theta_{\mathbf{p}}}{2} u_{\mathbf{p},s},$$

$$F_{\mathbf{p},s}^2 = u_{\mathbf{p}} \cos\frac{\theta_{\mathbf{p}}}{2} v_{\mathbf{p},s} + v_{\mathbf{p}} \sin\frac{\theta_{\mathbf{p}}}{2} u_{\mathbf{p},s},$$

$$F_{\mathbf{p},s}^3 = u_{\mathbf{p}} \sin\frac{\theta_{\mathbf{p}}}{2} u_{\mathbf{p},s} + v_{\mathbf{p}} \cos\frac{\theta_{\mathbf{p}}}{2} v_{\mathbf{p},s},$$

$$F_{\mathbf{p},s}^4 = u_{\mathbf{p}} \cos\frac{\theta_{\mathbf{p}}}{2} u_{\mathbf{p},s} - v_{\mathbf{p}} \sin\frac{\theta_{\mathbf{p}}}{2} v_{\mathbf{p},s}.$$

Simple algebraic manipulation gives

$$B(\mathbf{p}) = \partial_{p_y} \phi_{\mathbf{p}} \partial_{p_x} F_{\mathbf{p}} - \partial_{p_x} \phi_{\mathbf{p}} \partial_{p_y} F_{\mathbf{p}} = -\frac{\mathbf{p}}{|\mathbf{p}|^2} \cdot \nabla F_{\mathbf{p}} \quad (10)$$

and $F_{\mathbf{p}} = \sum_{\alpha=1,4,s} s(F_{\mathbf{p},s}^{\alpha})^2$. $\mathcal{N}$ can now be easily obtained as

$$\mathcal{N} = F_{\mathbf{0}} = v_{\mathbf{0},+}^2 = \theta(h - h_c). \quad (11)$$

On the other hand, from Eq. (10) and using integral by parts, $\mathcal{N}$ can also be given as $\mathcal{N} = 1/2\pi \int d^2\mathbf{p} \nabla \cdot (\mathbf{p}/|\mathbf{p}|^2) F_{\mathbf{p}} = \int d^2\mathbf{p} \delta(\mathbf{p}) F_{\mathbf{p}} = F_{\mathbf{0}}$. This analytical result implies that: (i) The singular behavior of Berry curvature comes solely from SOC in terms of the phase factor $\varphi_{\mathbf{p}}$ of the vortexlike solution of the BdG Hamiltonian; and (ii) it is the sudden change of $v_{\mathbf{0},+}^2$ instead of SOC that explicitly determines the topological phase transition. Consequently, there is a sudden change of the ground state wave function associated with the component of triplet pairing of the quasiparticles denoted by $\beta_{\mathbf{p},+}$ at zero momentum. This is also reflected in the momentum distribution investigated in [28] as can be seen from Eq. (3) that $\mathcal{E}_{\mathbf{0},+}/E_{\mathbf{0},+} = \mathrm{sgn}(h_c - h)$ and conclusively proves that the topological phase transition is directly related to the momentum distribution which can be readily measured in cold atom experiments. Finally, as expected, this sudden change of the ground state wave function also leads to divergent behavior of FS at $h = h_c$. The numerical result of FS is presented in Fig. 2 together with the momentum distribution of the total particle numbers Eq. (3) on both sides of the critical point where the parameters are chosen as $E_b/E_F = 0.5$ and $m\lambda/k_F = 1$ such that no first-order phase transition happens.

In order to obtain the critical behavior of FS around the transition point, we use another equivalent form of FS [29]: $\chi(h) = \sum_m |\langle m|H_I|G\rangle|^2/(E_m - E_g)^2$, where $|m\rangle$ denotes the excited state and E_m the excitation energy. Here $H_I = \partial_h H$ can be considered as the driving term and $\chi(h)$ is directly related to the dynamical response of the system [24]. After direct but lengthy calculations, we obtain FS as functions of h:

$$\chi(h) = \sum_{\mathbf{p}>0} \left[\frac{M_{\mathbf{p},+}^2}{E_{\mathbf{p},+}^2} + 2\frac{M_{\mathbf{p},0}^2}{(E_{\mathbf{p},+} + E_{\mathbf{p},-})^2} + \frac{M_{\mathbf{p},-}^2}{E_{\mathbf{p},-}^2} \right], \quad (12)$$

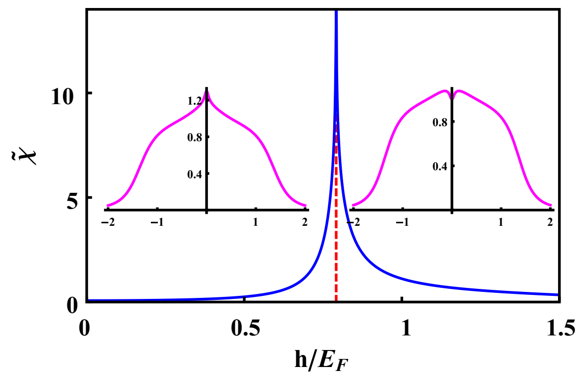

FIG. 2. (Color online) Fidelity susceptibility (in arbitrary dimension) as functions of Zeeman field with $E_b/E_F = 0.5$ and $m\lambda/k_F = 1$. The vertical red dashed line denotes the critical Zeeman field $h_c = 0.792$. The two inserted figures correspond to the momentum distribution of the total particle numbers shown in Eq. (3) as a function of p_x with $p_y = 0$.

where matrix elements $M_{\mathbf{p},s}$ and $M_{\mathbf{p},0}$ are given in Appendix A.

From Eq. (12) it is clear that the divergent behavior solely comes from the first term where $E_{\mathbf{p},+} = 0$ at the transition point around the Fermi point $p_x = p_y = 0$. Besides, since the gap and chemical potential vary continuously across the phase transition, derivatives of gap and chemical potential with respect to h are irrelevant constants for the critical behavior of FS. Therefore, we can consider Δ, μ, and h as independent controlling parameters and $H_I = \partial_h H = -\sigma_z$, which indicates that FS is directly related to the spin-spin correlation function and significantly simplifies the matrix elements M_s and M_0. Close to the critical Zeeman coupling h_c, the asymptotic form of $M_{\mathbf{p},+}$ and the excitation spectrum $E_{\mathbf{p},+}$ around the Fermi points can be given as $M_{\mathbf{p},+} \simeq -(\Delta\lambda/2h_c)p_\perp/E_{\mathbf{p},+}$ and $E^2_{\mathbf{p},+} = \Delta^2\lambda^2 p^2_\perp/h^2_c + |h - h_c|^2$, respectively. Substituting these asymptotic results into Eq. (12), we obtain the critical behavior of FS as

$$\chi(h) \propto -\ln|h - h_c|, \tag{13}$$

with details given in Appendix B.

As a byproduct, we conclude that existence of gapless excitation of the system does not necessarily means divergence of FS and phase transition. For example, in the absence of SOC, the Hamiltonian still supports gapless excitation spectrum when $h = h_c$ [30,31]. However, it does not lead to a divergent behavior of FS which can be understood in the following aspects. First, $\partial_h H_0$ in H_I commutes with total Hamiltonian without SOC, therefore it does not contribute to FS. Second, it is easy to show that $\partial_h H_{\mathrm{BCS}}$ only supports pairing excitations with opposite spins, therefore $M^2_{\mathbf{p},\pm} = 0$ and only the second term in Eq. (12) is not zero. Finally, without SOC, the excitation spectrum takes the following form: $E_{\mathbf{p},s} = \sqrt{(\mathbf{p}^2/2m - \mu)^2 + \Delta^2} - sh$ and the combination $E_{\mathbf{p},+} + E_{\mathbf{p},-} = 2\sqrt{(\mathbf{p}^2/2m - \mu)^2 + \Delta^2}$ is always gapped. Therefore, without SOC, the gapless nature of the excitation spectrum does not manifest itself by causing divergence of FS and therefore no continuous phase transition.

V. CONCLUSION

We investigate the ground-state properties of a pairing system in the presence of both SOC and Zeeman coupling that supports nontrivial topological order. In particular, we obtain the analytical result for the topological invariant which directly relates the topological phase transition with a sudden change of the BCS-type ground state wave function at zero momentum. Furthermore, it conclusively demonstrates that the topological phase transition can be determined by measuring the momentum distribution in cold atomic experiments. Generalization of this method of evaluating the topological invariant to higher dimensions and lattice situations will be of great interest. Last but not least, in the absence of Zeeman field without phase transitions, FS shows some features that are not revealed by other thermodynamic quantities in the BCS-BEC crossover induced by SOC.

ACKNOWLEDGMENTS

We acknowledge helpful discussions with Professor Gordon Baym, Anthony J. Leggett, and S. J. Gu. This work has been supported by the National Natural Science Foundation of China under Grant 51331006, 11204321, the National Basic Research Program No. 2010CB934603, the Ministry of Science and Technology of China and in part by NSF Grants PHY09-69790 and PHY13-05891. K. Z. Zhou also acknowledges the China Scholarship Council for supporting his visit to UIUC where part of this work is carried out.

APPENDIX A: DERIVATION OF EQ. (12)

Fidelity susceptibility are defined as

$$\chi = \sum_m \frac{|\langle m|H_I|G\rangle|^2}{(E_m - E_g)^2}, \tag{A1}$$

with $|G\rangle$ and $|m\rangle$ denoting the ground and excited state, respectively. Here $H_I = \partial H/\partial h = \sum_{\mathbf{p}>0} \Phi^\dagger_{\mathbf{p}} \partial_h H_{\mathrm{BdG}}(\mathbf{p})\Phi_{\mathbf{p}} + C$, with $\partial_h H_{\mathrm{BdG}}(\mathbf{p})$ being given as

$$\partial_h H_{\mathrm{BdG}}(\mathbf{p}) = \begin{bmatrix} -\partial_h\mu - 1 & 0 & 0 & -\partial_h\Delta \\ 0 & -\partial_h\mu + 1 & \partial_h\Delta & 0 \\ 0 & \partial_h\Delta & \partial_h\mu + 1 & 0 \\ -\partial_h\Delta & 0 & 0 & \partial_h\mu - 1 \end{bmatrix} \tag{A2}$$

and C being a constant. Since only excited states accounts for the summation in Eq. (A1), C does not contribute to χ and can be neglected in the following discussion.

Using the Bogoliubov transformation described in the main text, H_I can be rewritten in terms of the quasiparticle operator $\alpha_{\mathbf{p},s} = u_{\mathbf{p},s}\beta_{\mathbf{p},s} - e^{is\varphi_{\mathbf{p}}} v_{\mathbf{p},s}\beta_{-\mathbf{p},s}$:

$$H_I = \sum_{\mathbf{p}>0} [\alpha_{-\mathbf{p},+} \quad \alpha^\dagger_{\mathbf{p},+} \quad \alpha_{-\mathbf{p},-} \quad \alpha^\dagger_{\mathbf{p},-}] M' \begin{bmatrix} \alpha^\dagger_{-\mathbf{p},+} \\ \alpha_{\mathbf{p},+} \\ \alpha^\dagger_{-\mathbf{p},-} \\ \alpha_{\mathbf{p},-} \end{bmatrix}, \tag{A3}$$

with M' being the matrix elements. Substitution of Eq. (A3) into Eq. (A1) and using the fact that $\alpha_{\mathbf{p},s}|G\rangle = 0$, one can obtain Eq. (12),

$$\chi(h) = \sum_{\mathbf{p}>0} \left[\frac{M_{\mathbf{p},+}^2}{E_{\mathbf{p},+}^2} + 2\frac{M_{\mathbf{p},0}^2}{(E_{\mathbf{p},+}+E_{\mathbf{p},-})^2} + \frac{M_{\mathbf{p},-}^2}{E_{\mathbf{p},-}^2} \right], \quad \text{(A4)}$$

with matrix elements $M_{\mathbf{p},+}$, $M_{\mathbf{p},-}$, and $M_{\mathbf{p},0}$ are given as

$$\begin{aligned} M_{\mathbf{p},+} &= \partial_h \mu \left(v_{\mathbf{p}}^2 - u_{\mathbf{p}}^2\right) u_{\mathbf{p},+} v_{\mathbf{p},+} \\ &\quad + (1 - 2u_{\mathbf{p}} v_{\mathbf{p}} \partial_h \Delta) u_{\mathbf{p},+} v_{\mathbf{p},+} \cos\theta_{\mathbf{p}} \\ &\quad + \left(u_{\mathbf{p}} v_{\mathbf{p}} - \frac{\partial_h \Delta}{2}\right) \sin\theta_{\mathbf{p},}\left(u_{\mathbf{p},+}^2 - v_{\mathbf{p},+}^2\right), \end{aligned} \quad \text{(A5)}$$

$$\begin{aligned} M_{\mathbf{p},-} &= \left(u_{\mathbf{p}}^2 - v_{\mathbf{p}}^2\right)(u_{\mathbf{p},+} v_{\mathbf{p},-} - u_{\mathbf{p},-} v_{\mathbf{p},+}) \sin\theta_{\mathbf{p}} \\ &\quad + \left[2u_{\mathbf{p}} v_{\mathbf{p}} \partial_h \mu - \partial_h \Delta \left(u_{\mathbf{p}}^2 - v_{\mathbf{p}}^2\right) \cos\theta_{\mathbf{p}}\right] \\ &\quad \times (u_{\mathbf{p},+} u_{\mathbf{p},-} + v_{\mathbf{p},+} v_{\mathbf{p},-}), \end{aligned} \quad \text{(A6)}$$

$$\begin{aligned} M_{\mathbf{p},0} &= \partial_h \mu (v^2 - u^2) u_- v_- - (1 + 2uv\partial_h \Delta) u_- v_- \cos\theta \\ &\quad - \left(uv + \frac{\partial_h \Delta}{2}\right) \sin\theta (u_-^2 - v_-^2). \end{aligned} \quad \text{(A7)}$$

APPENDIX B: DERIVATION OF EQ. (13)

At the critical point $h = h_c$, the excitation spectrum $E_{\mathbf{p},+} = 0$ at $\mathbf{p} = \mathbf{0}$, while $E_{\mathbf{p},-}$ is always gapped. From Eq. (A4) it is clear that the integration is well behaved in the ultraviolet limit. In the infrared limit, only the first term may cause divergent behavior of $\chi(h_c)$ because of the gapless excitation spectrum in the denominator in the $\mathbf{p} \to \mathbf{0}$ regime.

In order to obtain the asymptotic form of the excitation spectrum, we investigate the following identity:

$$\begin{aligned} E_{\mathbf{p},+}^2 E_{\mathbf{p},-}^2 &= \left(\varepsilon_{\mathbf{p}}^2 + \Delta^2 + h^2 + |\Gamma_{\mathbf{p}}|^2\right)^2 \\ &\quad - 4\left(\varepsilon_{\mathbf{p}}^2 h^2 + \varepsilon_{\mathbf{p}}^2 |\Gamma|^2 + \Delta^2 h^2\right). \end{aligned}$$

Around the critical point $h = h_c + \eta$, with η being small, Taylor expansion of the above identity to the lowest (second) order in $p_\perp$ and η, we obtain

$$E_{\mathbf{p},+}^2 E_{\mathbf{p},-}^2 \to 4\Delta^2 \lambda^2 p_\perp^2 + 4h_c^2 \eta^2. \quad \text{(B1)}$$

Since $E_{\mathbf{p},-}$ is always gapped, in the zeroth order,

$$E_{\mathbf{p},-}^2 \to (\sqrt{\mu^2 + \Delta^2} + h)^2 = 4h_c^2. \quad \text{(B2)}$$

Together with Eqs. (B1) and (B2), we obtain the asymptotic form of the excitation spectrum $E_{\mathbf{p},+}^2$:

$$E_{\mathbf{p},+}^2 = \frac{\Delta^2 \lambda^2}{h_c^2} p_\perp^2 + \eta^2. \quad \text{(B3)}$$

As have been stated in the main text, for the topological phase transition considered in this paper, gap parameter Δ and chemical potential μ as functions of Zeeman field h are continuous and their derivatives (appearing in the enumerator) are irrelevant to the divergent behavior of $\chi(h)$. Therefore, matrix element $M_{\mathbf{p},+}$ can be reduced to

$$\begin{aligned} M_{\mathbf{p},+} &= u_{\mathbf{p},+} v_{\mathbf{p},+} \cos\theta_{\mathbf{p}} + u_{\mathbf{p}} v_{\mathbf{p}} \sin\theta_{\mathbf{p},}\left(u_{\mathbf{p},+}^2 - v_{\mathbf{p},+}^2\right) \\ &= \frac{1}{2} \frac{\Delta \sin\theta_{\mathbf{p}} \cos\theta_{\mathbf{p}}}{E_{\mathbf{p},+}} + \frac{1}{2} \frac{\Delta |\cos\theta_{\mathbf{p}}| \sin\theta_{\mathbf{p}}}{E_{\mathbf{p}}} \frac{\mathcal{E}_{\mathbf{p},+}}{E_{\mathbf{p},+}} \\ &= \frac{1}{2} \frac{\Delta \sin\theta_{\mathbf{p}} |\cos\theta_{\mathbf{p}}|}{E_{\mathbf{p},+}} \left(\frac{\mathcal{E}_{\mathbf{p},+}}{E_{\mathbf{p}}} - 1\right). \end{aligned} \quad \text{(B4)}$$

Around the critical point h_c, $\sin\theta_{\mathbf{p}} \propto p_\perp$, $|\cos\theta_{\mathbf{p}}| = 1$, and $\mathcal{E}_{\mathbf{p},+} \propto p_\perp$. Therefore, at first order in $p_\perp$, $M_{\mathbf{p},+}$ can be approximated as

$$M_{\mathbf{p},+} \to -\frac{1}{2} \frac{\Delta \sin\theta}{E_{\mathbf{p},+}} = -\frac{1}{2} \frac{\Delta \lambda p_\perp}{E_{\mathbf{p},+} h_c}.$$

Together with Eq. (B3), we obtain the asymptotic form of $M_{\mathbf{p},+}$ as

$$M_{\mathbf{p},+} \to -\frac{1}{2} \frac{\Delta \lambda p_\perp}{\sqrt{\frac{\Delta^2 \lambda^2}{h_c^2} p_\perp^2 + |h - h_c|^2} h_c}. \quad \text{(B5)}$$

Substituting Eqs. (B5) and (B3) into the first term in Eq. (A4), we obtain

$$\chi(h) = \frac{1}{4} \sum_{\mathbf{p}>0} \frac{\Delta^2 \lambda^2 p_\perp^2}{h_c \left(\frac{\Delta^2 \lambda^2}{h_c^2} p_\perp^2 + |h - h_c|^2\right)^2}.$$

As usual, summation over momentum is replaced by integration in the continuum limit and we only consider the infrared regime of the momentum integration which causes divergence of $\chi(h)$. Based on these assumptions, we obtain

$$\begin{aligned} \chi(h) &\propto \int_0 p_\perp dp_\perp \frac{\Delta^2 \lambda^2 p_\perp^2}{h_c^2 \left(\frac{\Delta^2 \lambda^2}{h_c^2} p_\perp^2 + |h - h_c|^2\right)^2} \\ &\propto \int_0 dx \frac{\frac{\Delta^2 \lambda^2}{h_c^2} x}{\left(\frac{\Delta^2 \lambda^2}{h_c^2} x + |h - h_c|^2\right)^2} \\ &\propto \int dx \frac{x}{(x + |h - h_c|^2)^2} \simeq -1 - \ln|h - h_c|^2 \\ &\propto -\ln|h - h_c|, \end{aligned} \quad \text{(B6)}$$

where we neglect the exact value of the coefficient which in general depends on the derivative of Δ and μ with respect to h.

Finally, we conclude that the terms depending on the derivative of Δ and μ with respect to h does not affect the logarithmic divergence of χ. First, for $p_\perp \to 0$, $v_{\mathbf{p}}^2 - u_{\mathbf{p}}^2$, $1 - 2u_{\mathbf{p}} v_{\mathbf{p}} \partial_h \Delta$, $\cos\theta_{\mathbf{p}}$, and $u_{\mathbf{p}} v_{\mathbf{p}} - \frac{\partial_h \Delta}{2}$ are constants and $u_{\mathbf{p},+} v_{\mathbf{p},+} \propto p_\perp / E_{\mathbf{p},+}$ and $\sin\theta_{\mathbf{p}} \propto p_\perp$. Second, the third term is second order in $p_\perp$ as can be seen from Eq. (B4). Therefore, in the lowest order in $p_\perp$, $M_{\mathbf{p},+} \propto p_\perp / E_{\mathbf{p},+}$, which leads to the logarithmic divergence of $\chi(h)$ as can be seen from the first line of Eq. (B6).

[1] C. Nayak *et al.*, Rev. Mod. Phys. **80**, 1083 (2008).
[2] X. L. Qi and S. C. Zhang, Rev. Mod. Phys. **83**, 1057 (2011).
[3] B. A. Bernevig and T. L. Hughes, *Topological Insulators and Topological Superconductors* (Princeton University Press, Princeton, NJ, 2013).
[4] P. Ghosh, J. D. Sau, S. Tewari, and S. Das Sarma, Phys. Rev. B **82**, 184525 (2010); J. D. Sau, S. Tewari, R. M. Lutchyn, T. D. Stanescu, and S. Das Sarma, *ibid.* **82**, 214509 (2010).
[5] Y.-J. Lin, K. Jiménez-García, and I. B. Spielman, Nature (London) **471**, 83 (2011); R. A. Williams, M. C. Beeler, L. J. LeBlanc, K. Jiménez-García, and I. B. Spielman, Phys. Rev. Lett. **111**, 095301 (2013).
[6] P. Wang, Z.-Q. Yu, Z. Fu, J. Miao, L. Huang, S. Chai, H. Zhai, and J. Zhang, Phys. Rev. Lett. **109**, 095301 (2012).
[7] L. W. Cheuk, A. T. Sommer, Z. Hadzibabic, T. Yefsah, W. S. Bakr, and M. W. Zwierlein, Phys. Rev. Lett. **109**, 095302 (2012).
[8] J. Dalibard, F. Gerbier, G. Juzeliūnas, and P. Öhberg, Rev. Mod. Phys. **83**, 1523 (2011).
[9] N. Goldman, G. Juzeliunas, P. Ohberg, and I. B. Spielman, arXiv:1308.6533.
[10] K. Martiyanov, V. Makhalov, and A. Turlapov, Phys. Rev. Lett. **105**, 030404 (2010).
[11] B. Fröhlich, M. Feld, E. Vogt, M. Koschorreck, W. Zwerger, and M. Köhl, Phys. Rev. Lett. **106**, 105301 (2011).
[12] M. Feld *et al.*, Nature (London) **480**, 75 (2011).
[13] J. P. Vyasanakere, S. Zhang, and V. B. Shenoy, Phys. Rev. B **84**, 014512 (2011); J. P. Vyasanakere and V. B. Shenoy, Phys. Rev. A **86**, 053617 (2012).
[14] H. Hu, L. Jiang, X. J. Liu, and Han Pu, Phys. Rev. Lett. **107**, 195304 (2011).
[15] Z. Q. Yu and H. Zhai, Phys. Rev. Lett. **107**, 195305 (2011).
[16] L. Han and C. A. R. Sá de Melo, Phys. Rev. A **85**, 011606(R) (2012).
[17] L. Y. He and X. G. Huang, Phys. Rev. Lett. **108**, 145302 (2012).
[18] L. Dell'Anna, G. Mazzarella, and L. Salasnich, Phys. Rev. A **86**, 053632 (2012).
[19] M. Iskin and A. L. Subasi, Phys. Rev. Lett. **107**, 050402 (2011).
[20] M. Gong, S. Tewari, and C. W. Zhang, Phys. Rev. Lett. **107**, 195303 (2011).
[21] F. Wu, G. C. Guo, W. Zhang, and W. Yi, Phys. Rev. Lett. **110**, 110401 (2013).
[22] J. N. Zhang, Y. H. Chan, and L. M. Duan, arXiv:1110.2241.
[23] G. E. Volovik, *The Universe in a Helium Droplet* (Clarendon, Oxford, 2003).
[24] S.-J. Gu, Int. J. Mod. Phys. B **24**, 4371 (2008).
[25] Y. A. Bychkov and E. I. Rashba, J. Phys. C **17**, 6039 (1984); L. P. Gor'kov and E. I. Rashba, Phys. Rev. Lett. **87**, 037004 (2001).
[26] A. Khan and P. Pieri, Phys. Rev. A **80**, 012303 (2009).
[27] K. Z. Zhou and Z. D. Zhang, Phys. Rev. Lett. **108**, 025301 (2012).
[28] J. Zhou, W. Zhang, and W. Yi, Phys. Rev. A **84**, 063603 (2011).
[29] W. L. You, Y. W. Li, and S. J. Gu, Phys. Rev. E **76**, 022101 (2007); P. Zanardi, P. Giorda, and M. Cozzini, Phys. Rev. Lett. **99**, 100603 (2007).
[30] H. Hu and X. J. Liu, Phys. Rev. A **73**, 051603(R) (2006).
[31] M. Mannarelli, G. Nardulli, and M. Ruggieri, Phys. Rev. A **74**, 033606 (2006).